2D MODELLING OF GROUNDWATER FLOW USING FINITE ELEMENT METHOD IN AN OBJECT-ORIENTED APPROACH

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Abstract
A finite element numerical solution is presented for groundwater models discretised by triangular element grid. An object-oriented approach is considered for implementing the models. Thus, a Java program is developed for this purpose to obtain primarily the unknown hydraulic heads at the nodes of triangular elements, provided boundary conditions are given. By analysis of the hydraulic head solution results, one can easily obtain the flow direction vectors, flow velocities and flow rates in different directions.

Introduction
For mining excavations as well as many other engineering operations, groundwater situation in the area under operation should be characterised and considered in the design of the operation. For this, modelling of groundwater flow is often necessary to estimate the groundwater characteristics in the area. Groundwater flow equation in simple form corresponds to the Laplace's equation, which represent a potential problem. In this paper, a numerical solution for the groundwater flow equation using finite element method is obtained for a practical case by implementing the model in the object-oriented programming language Java.

Formulation of the Problem
Assuming an anisotropic non-horizontal layered soil represented in two dimensions x and z of a Cartesian coordinate system as shown in Figure 1, we can write Darcy's filtration law in the directions ξ and η (see Fig. 1) in the following forms:

$$v_\xi = -k_\xi \frac{\partial h}{\partial \xi} \qquad \text{and} \qquad v_\eta = -k_\eta \frac{\partial h}{\partial \eta}, \tag{1}$$

where v is the flow velocity, h is height of water free surface or level (piezometric head) and k is transmissivity (hydraulic conductivity). In matrix notation, equations (1) are written as

$$\begin{bmatrix} v_\xi \\ v_\eta \end{bmatrix} = -\begin{bmatrix} k_\xi & 0 \\ 0 & k_\eta \end{bmatrix} \begin{bmatrix} \frac{\partial h}{\partial \xi} \\ \frac{\partial h}{\partial \eta} \end{bmatrix}. \tag{2}$$

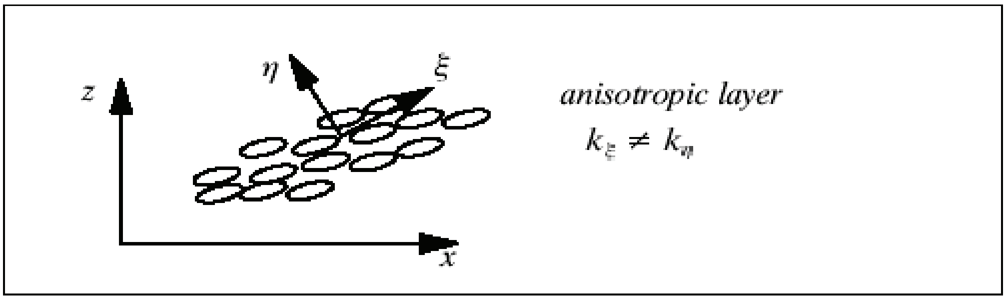

Figure 1. An anisotropic non-horizontal layered soil 2D model.

As computation has to be performed within a global x - z coordinate system, a coordinate transformation has to be made from layer coordinates to global one. For this purpose a rotation matrix τ, defined in the following, is used for this transformation:

$$\tau = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \tag{3}$$

where θ is the angle between horizontal axes of the two coordinate systems (i.e. axes x and ξ). Finally, for the anisotropic soil, we obtain the transmissivity matrix $\underline{k}$ in the global coordinate system as:

$$\underline{k} = \begin{bmatrix} k_{xx} & k_{xz} \\ k_{zx} & k_{zz} \end{bmatrix}. \tag{4}$$

The matrix $\underline{k}$ is symmetric. Thus, it holds $k_{xz} = k_{zx}$.

Writing the Darcy's law in the global x - z coordinate system for this case, i.e.

$$v_x = -k_{xx}\frac{\partial h}{\partial x} - k_{xz}\frac{\partial h}{\partial z} \qquad \text{and} \qquad v_z = -k_{zx}\frac{\partial h}{\partial x} - k_{zz}\frac{\partial h}{\partial z}, \tag{5}$$

and combining these equations with the equation for mass conservation (continuity equation), given by

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0, \tag{6}$$

we finally get the Poisson's equation for groundwater flow as

$$k_{xx}\left(\frac{\partial^2 h}{\partial x^2}\right) + 2k_{xz}\left(\frac{\partial^2 h}{\partial x \partial z}\right) + k_{zz}\left(\frac{\partial^2 h}{\partial z^2}\right) = 0. \tag{7}$$

For an isotropic soil, $k_{xx} = k_{zz} = k$ and $k_{xz} = k_{zx} = 0$, thus equation (7) changes to the Laplace's equation, i.e.

$$k\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2}\right) = 0 \qquad \text{or} \qquad k\,\Delta(h) = 0, \tag{8}$$

where Δ denotes the Laplace's operator. This formulation is for the situation when the constant transmissivity k holds for the entire domain of solution. For finite element solutions, which will be discussed in this paper, it is generally considered that the transmissivity may change from element to element, but it is constant within each individual element. The more general approach is obtained by assuming the transmissivity k being a function of space. In this case, the Laplace's equation is written in the following form:

$$\frac{\partial^2}{\partial x^2}(k\,h) + \frac{\partial^2}{\partial z^2}(k\,h) = 0 \qquad \text{or} \qquad \Delta(kh) = 0. \tag{9}$$

Analytical solutions to the Laplace's equation may be obtained by introducing a potential function $\phi = -kh$, the derivatives of which give the velocities v_x and v_z, i.e.

$$v_x = \frac{\partial \phi}{\partial x} = -k\frac{\partial h}{\partial x} \qquad \text{and} \qquad v_z = \frac{\partial \phi}{\partial z} = -k\frac{\partial h}{\partial z}, \tag{10}$$

which are always orthogonal to the potential function. Alternatively considering the mass conservation equation given by equation (6), we can use a special function, the stream function ψ defined in the following, for the solution of the above differential equations:

$$v_x = \frac{\partial \psi}{\partial z} \qquad \text{and} \qquad v_z = -\frac{\partial \psi}{\partial x}. \tag{11}$$

Streamlines (a set of stream functions $\psi_i = const.$ given within the domain of solution) are also orthogonal to the equi-potential function lines. However, at any point of the stream function $\psi_i = const.$ the flow velocity is tangential to the function. Considering velocities given by equation (11), we can write equation (5) in the following matrix form:

$$\begin{bmatrix} \dfrac{\partial \psi}{\partial z} \\ -\dfrac{\partial \psi}{\partial x} \end{bmatrix} = -\begin{bmatrix} k_{xx} & k_{xz} \\ k_{zx} & k_{zz} \end{bmatrix}\begin{bmatrix} \dfrac{\partial h}{\partial x} \\ \dfrac{\partial h}{\partial z} \end{bmatrix}. \tag{12}$$

This matrix equation should be inverted in order to solve for the water head (potential) function, i.e.

$$\begin{bmatrix} \dfrac{\partial h}{\partial x} \\ \dfrac{\partial h}{\partial z} \end{bmatrix} = -\begin{bmatrix} k_{xx} & k_{xz} \\ k_{zx} & k_{zz} \end{bmatrix}^{-1}\begin{bmatrix} \dfrac{\partial \psi}{\partial z} \\ -\dfrac{\partial \psi}{\partial x} \end{bmatrix}. \tag{13}$$

For the Laplace's equation case (isotropic soil), a potential Π exists and it is given by the following surface integral equation:

$$\Pi = \frac{1}{2}\iint\left[k\left(\frac{\partial h}{\partial x}\right)^2 + k\left(\frac{\partial h}{\partial z}\right)^2\right]dxdz\ . \qquad (14)$$

Similarly for the Poisson's equation case (anisotropic soil), the potential is expressed by

$$\Pi = \frac{1}{2}\iint\left[k_{xx}\left(\frac{\partial h}{\partial x}\right)^2 + 2k_{xz}\frac{\partial h}{\partial x}\frac{\partial h}{\partial z} + k_{zz}\left(\frac{\partial h}{\partial z}\right)^2\right]dxdz\ . \qquad (15)$$

Finite Element Scheme for the Numerical Solution of the Problem

We consider a typical groundwater problem as illustrated in Figure 2. For modelling such a system, it should be discretised and boundary conditions should also be prescribed. For discretisation of the system in the finite element scheme, we use a triangular grid as shown in Figure 2.

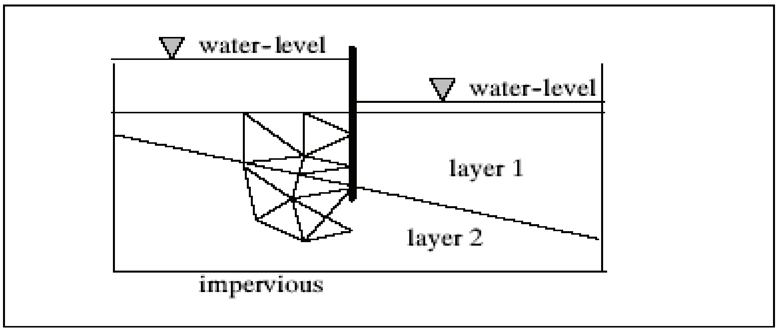

Figure 2. The groundwater model discretised by triangular finite elements.

The hydraulic head $h(x,z)$ at the nodes of the triangular grid with *n* nodes and *m* elements are to be determined by the finite element solution presented here. The water heads at the nodes are represented by $\bar{h}_i; i = 0,1,2,...,n-1$. They may be assembled within the vector matrix $\underline{H}$ where $\underline{H}^T = \left[\bar{h}_0 \ \ \bar{h}_1 \ \ \bar{h}_2 \ ... \ \bar{h}_{n-1}\right]$. Within the elements the potential function can be expressed by a linear approximation function in neutral (area) coordinates. The approximation head function h for each triangular element, represented in Figure 3, can also be defined linearly, i.e.

$$h = \sum_{l=i,j,k} \lambda_l \bar{h}_l\ , \qquad (16)$$

where λ_l and $\bar{h}_l$ denotes, respectively, the neutral coordinates and hydraulic heads for the 3 nodes (represented by i, j, k) existing in each triangular element (see Fig. 3). As the potential is based on the first derivatives of the state (head) function, the derivatives of the approximation (head) function, given by the following expression, have to be calculated:

$$\frac{\partial h(\lambda)}{\partial x} = \frac{\partial h}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial x} + \frac{\partial h}{\partial \lambda_j}\frac{\partial \lambda_j}{\partial x} + \frac{\partial h}{\partial \lambda_k}\frac{\partial \lambda_k}{\partial x}\ , \qquad (17)$$

which can be recited in the following form:

$$\frac{\partial h}{\partial x} = \frac{1}{2A}(\beta_i \bar{h}_i + \beta_j \bar{h}_j + \beta_k \bar{h}_k)\ , \qquad (18)$$

where A is the area of the triangle, and is defined by the following determinant:

$$A = \frac{1}{2}\det\begin{bmatrix} 1 & x_i & z_i \\ 1 & x_j & z_j \\ 1 & x_k & z_k \end{bmatrix} \quad \text{or} \quad 2A = \det\begin{bmatrix} 1 & x_i & z_i \\ 1 & x_j & z_j \\ 1 & x_k & z_k \end{bmatrix}. \qquad (19)$$

Similarly, the first derivatives of the approximation (head) function with respect to *z* is defined as

$$\frac{\partial h}{\partial z}=\frac{1}{2A}(\gamma_i\bar{h}_i+\gamma_j\bar{h}_j+\gamma_k\bar{h}_k)\,. \quad (20)$$

The transformation from neutral coordinates λ to Cartesian coordinates *x–z* within a triangular element is given by:

$$\lambda_i=\frac{1}{2A}(\alpha_i+\beta_i x+\gamma_i z),\ \ \lambda_j=\frac{1}{2A}(\alpha_j+\beta_j x+\gamma_j z)\ \text{ and }\ \lambda_k=\frac{1}{2A}(\alpha_k+\beta_k x+\gamma_k z)\,. \quad (21)$$

The data α , β and γ are computed by

$$\begin{array}{lll} \alpha_i=x_j z_k-x_k z_j & \beta_i=z_j-z_k & \gamma_i=x_k-x_j \\ \alpha_j=x_k z_i-x_i z_k & \beta_j=z_k-z_i & \gamma_j=x_i-x_k\,. \\ \alpha_k=x_i z_j-x_j z_i & \beta_k=z_i-z_j & \gamma_k=x_j-x_i \end{array} \quad (22)$$

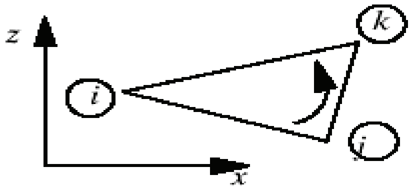

Figure 3. Each triangular element has three nodes *i*, *j* and *k* arranged in counter-clockwise direction.

The flow velocities v_x and v_z (or the flow rate, which is velocity multiplied by thickness *y* of vertical soil slice) are computed using the Darcy's law and applying the expressions for the first derivatives of the head functions, given by equation (17) or (18) and (20).

Using the Petrov-Galerkin method for the weighted residual approach in the finite element integral, and considering the transmissivity *k* being constant and also using linear approximation (head) functions in neutral (area) coordinates for each triangular element, we construct the following matrix equation for any triangular element with the nodes *i*, *j* and *k*:

$$\begin{bmatrix} k_{ii} & k_{ij} & k_{ik} \\ k_{ji} & k_{jj} & k_{jk} \\ k_{ki} & k_{kj} & k_{kk} \end{bmatrix}\begin{bmatrix} \bar{h}_i \\ \bar{h}_j \\ \bar{h}_k \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{or} \qquad \underline{k}\bar{\underline{h}}=0\,,$$

where $\underline{k}$ is the coefficient or transmissivity matrix that is generated on the element level. For the global system the transmissivity matrices from all elements have to be assembled, and as a result, the transmissivity coefficients for a node belonging to two or more elements are summed up to obtain one transmissivity value for the corresponding entry of the coefficient matrix in the global equation system. We can write the global equation system in general matrix form $\underline{K}\underline{H}=0$ where the coefficient or transmissivity matrix $\underline{K}$ is sparse, symmetric and positive definite, and will be of $n\times n$ size if the discretised groundwater modelling domain consists of n nodes. The vector matrix $\underline{H}$ includes unknown hydraulic heads at the nodes, i.e. $\underline{H}^T=\lfloor \bar{h}_0\ \ \bar{h}_1\ \ \bar{h}_2 \ldots \bar{h}_{n-1} \rfloor$. A solution for the unknown hydraulic heads at the nodes is obtained by prescribing boundary conditions. To obtain the solution the given hydraulic heads at nodes (boundary conditions) are multiplied by the corresponding elements of the coefficient matrix $\underline{K}$, and then the results of these multiplication are correspondingly subtracted from (and moved to) the right hand side of the equation system which contains the known information about the system. As a result, we obtain the non-homogeneous equation system $\underline{K}\underline{H}=\underline{R}$, where $\underline{R}$ is the vector matrix containing known values. Finally, by inverting this matrix equation system, we obtain the unknown values (hydraulic heads) of vector matrix $\underline{H}$, i.e. $\underline{H}=\underline{K}^{-1}\underline{R}$. As the matrix $\underline{K}$ is sparse and symmetric, we use a proper efficient fast solver for symmetric equation systems such as Preconditioned Conjugate Gradient (PCG) method or Cholesky algorithm to solve the equation system.

An object-oriented programming approach is considered for implementation of a model and obtaining the model solution because of its flexibility and less time demanding for future extension of the model. For this, a Java program has been developed to obtain the water heads at the nodes of triangular element discretised grid within the groundwater modelling domain in the finite element scheme.

Numerical Examples

Although we have implemented and obtained the results for different models, here only the results of a simplified model, shown in Figure 4, have been presented. The boundary conditions $\bar{h}_0 = \bar{h}_1 = 40\,\text{m}$ and $\bar{h}_{10} = \bar{h}_{11} = 20\,\text{m}$ are given. Also we have considered the edges of the elements along x and z directions to be equal to 5 m (i.e. $\Delta x = \Delta z = 5\,\text{m}$). As indicated in Figure 5, the following hydraulic heads at nodes 2 to 9 are obtained by running the finite element based Java program for the above simplified model: $\bar{h}_2 = 35.172\,\text{m}$, $\bar{h}_3 = 33.517\,\text{m}, \bar{h}_4 = 33.655\,\text{m}, \bar{h}_5 = 32.138\,\text{m}, \bar{h}_6 = 26.483\,\text{m}, \bar{h}_7 = 27.862\,\text{m}, \bar{h}_8 = 26.345\,\text{m}$ and $\bar{h}_9 = 24.828\,\text{m}$. Also, the flow direction vectors and hydraulic head contour lines have been shown in Figure 5. As can be seen, the water flows from high heads toward low heads in a semi-circular shape.

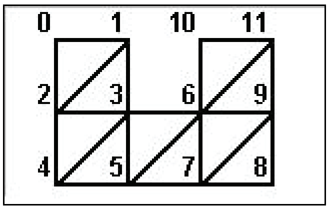

Figure 4. A simplified model with nodes numbered from 0 to 11 in the triangular element grid.

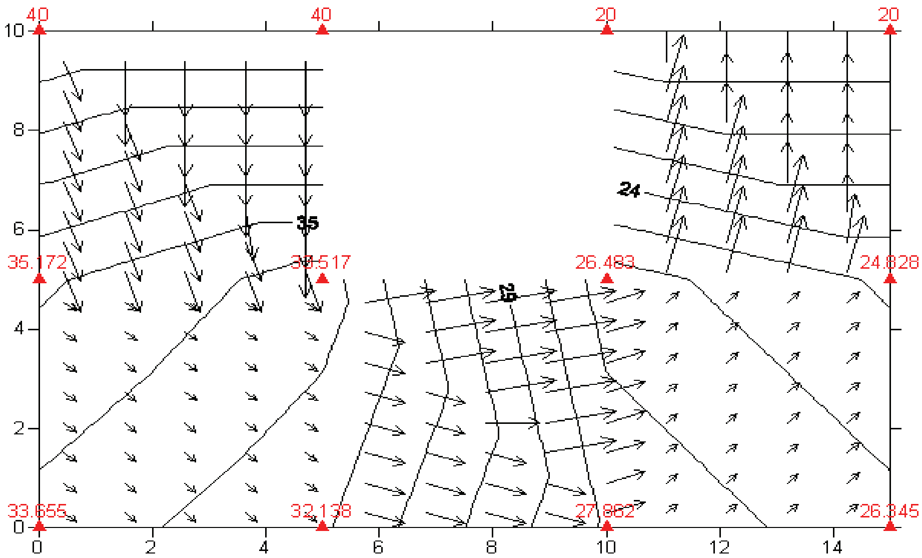

Figure 5. The flow direction vectors, hydraulic head values at nodes (indicated by small solid triangles) and hydraulic head contour lines for the simplified model shown in Figure 4.

Conclusions

A numerical solution for the groundwater flow equation using finite element scheme was obtained. A number of models discretised by triangular element grid were considered in this work, and a Java program was developed to obtain primarily unknown hydraulic heads at the nodes of triangular elements. Based on the results, the flow direction vectors and hydraulic head contour lines were shown for a model.

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