A Finite Difference Groundwater Modelling and Comparison of the Results with Those Obtained Using Finite element modelling approach

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Abstract

A finite difference modelling problem formulation with the help of a Taylor-series expansion is presented for groundwater models. The obtained differential groundwater equation is, then, solved using a suitable numerical solution algorithm based on the finite difference scheme over computational domains discretised by rectangular element grid. Defining boundary conditions, we apply the finite difference solution algorithm to obtain primarily the unknown hydraulic heads at the nodes of rectangular elements in the model. By analysis of the hydraulic head solution results, we can easily obtain the flow velocities at the nodes of rectangular elements. Although the implementation of the algorithm and obtaining the results have been carried out for various models, here only the results of a simple model will be presented. Finally, the results of the finite difference solution are compared with the results of a finite element modelling solution for the same model.

Key words: groundwater modelling, finite difference and finite element methods, boundary conditions, numerical solution, computational domain

Introduction

In recent years, numerical models have become indispensable in groundwater simulations, mainly for making predictions and improving process understanding. For many underground or open pit mining excavations as well as many other engineering operations, groundwater situation in the area under operation should be characterised and considered in the mining or operation design. To determine the groundwater characteristics in the area, modelling of groundwater flow is often needed. One of the most important numerical methods for modelling of groundwater flow is the finite difference method, which is used for building, and then, solving partial differential groundwater equations. In this paper, a finite difference formulation for groundwater flow modelling problem is derived, and then, a numerical solution for the problem based on the method is obtained. We consider a model, which can simulate groundwater situation in a mining site with a simple subsurface geology. Also, for the same model, the finite difference results are compared with the results of a finite element solution algorithm.

Formulation of the Finite Difference Modelling Problem

The finite difference method is a numerical method, which can be used for solving partial differential groundwater equations. The computational domain is discretised by rectangular cells (see Figure 1) although quadrilateral cells can also be used. For simplicity, we consider the cell lengths in the x and z directions to be constant and equal, i.e. $\Delta x = \Delta z$. The unknown variables are defined in the nodes which are placed at the centers of the cells or at the intersection points of cell boundaries (see Figure 1). To follow a unique law for the nodes, we consider them to be at the intersection points of cell boundaries throughout this paper. From the geometrical point of view, it is obvious that complex boundaries or complex inner structures can only be reproduced in a very simplified way by step functions.

The formulation of the finite difference modelling problem is basically carried out by substituting the differential functions by approximated values derived from Taylor-series expansions of the functions. The equations are then put together in an explicit or implicit way. In this method, by developing the derivatives of unknown functions with the help of Taylor-series expansions and taking into account initial and/or boundary conditions, we obtain the solutions to the problem (Hinkelmann 2005).

We consider groundwater situation in an area (e.g. a mining site), as shown in Figure 2. Groundwater flow is driven by pressure gradient. The pressure gradient is due to differences of water level or height (i.e. Δh in Figure 2). For such a flow, Darcy's filtration law in a simple form can be given by the following equation:

$$v = -K \nabla h , \quad (1)$$

where v is the Darcy or filter velocity, K is the hydraulic conductivity (or transmissivity) and h is the water level or height, which is also called the hydraulic or piezometric head. In equation (1), ∇h

or $grad\, h$ represents the gradient h and thus, for the case of one-dimensional (1-D) homogeneous flow, this equation is similpified into the following form:

$$v = -K \frac{\partial h}{\partial x} \quad (2)$$

For a steady groundwater flow, we have:

$$\nabla v = \frac{q_w}{\rho_w}, \qquad \text{or} \qquad \nabla v = q, \quad (3)$$

Where ∇v or $div\, v$ represents the divergence v, q_w is the flux of wetting phase (in terms of $Kg/m^3 s$), ρ_w is the density of water (in terms of Kg/m^3), and q is the specific discharge, which is the discharge over a volume of water (in terms of $m^{3/}s/m^3$ or $1/s$). Substituting equation (1) in equation (3), we obtain:

$$div\,(-K\ grad\ h) = q. \quad (4)$$

Generally, for the case of three-dimensional (3-D) inhomogeneous medium, the hydraulic conductivity K is defined by the following tensor, which is symmetric:

$$K = \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix}. \quad (5)$$

For the case of 1-D homogeneous medium, equation (4) is similpified into the following form:

$$-K \frac{\partial^2 h}{\partial x^2} = q. \quad (6)$$

For the solution of the differential equation (6) using the finite difference scheme, first we discretise the computational domain, as shown in Figure 1. Considering the Taylor-series expansions for the unknown hydraulic head functions h_{i+1} and h_{i-1}, given by equations (7) and (8), and then, truncating the expansions to the third term, and adding these two truncated expansions together, we can then extract the second derivative of hydraulic head with respect to x:

$$h_{i+1} = h_i + \Delta x \frac{\partial h}{\partial x} + \frac{(\Delta x)^2}{2!} \frac{\partial^2 h}{\partial x^2} + \ldots \quad (7)$$

$$h_{i-1} = h_i - \Delta x \frac{\partial h}{\partial x} + \frac{(\Delta x)^2}{2!} \frac{\partial^2 h}{\partial x^2} - \ldots \quad (8)$$

$$\frac{\partial^2 h}{\partial x^2} = \frac{h_{i+1} - 2h_i + h_{i-1}}{(\Delta x)^2}. \quad (9)$$

Substituting equation (9) in equation (6), and then extracting h_i from the resultant equation, we obtain:

$$h_i = \frac{h_{i+1} + h_{i-1}}{2} + \frac{q_i (\Delta x)^2}{2K}. \quad (10)$$

Defining initial and/or boundary conditions, we solve the problem by obtaining the hydraulic heads h at the nodes. For fast computation by computer, the problem can be written in the matrix equation form of $AH = b$, where the coefficients matrix A and vector matrix b are known, and the vector matrix H, comprising the unknown hydraulic heads h at the nodes, are obtained by solving this matrix equation (i.e. $H = A^{-1}b$, where A^{-1} is the inverse matrix A). It should be noted that matrix A must be symmetric, positive definite, sparce, and its diagonal terms must be positive as well.

***Figure 1** Space discretisation of the modelling domain in the finite difference method*

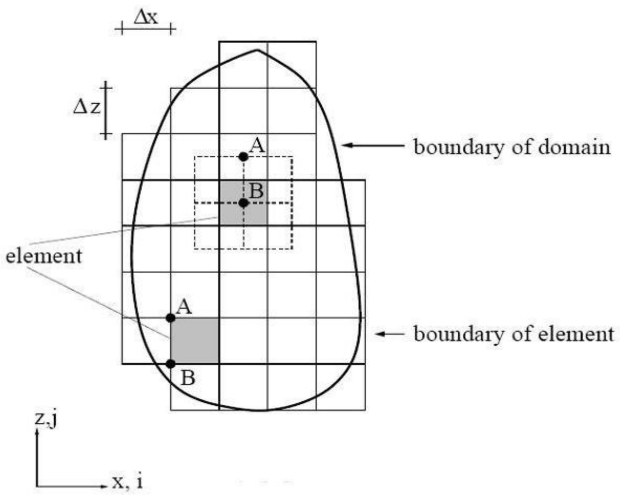
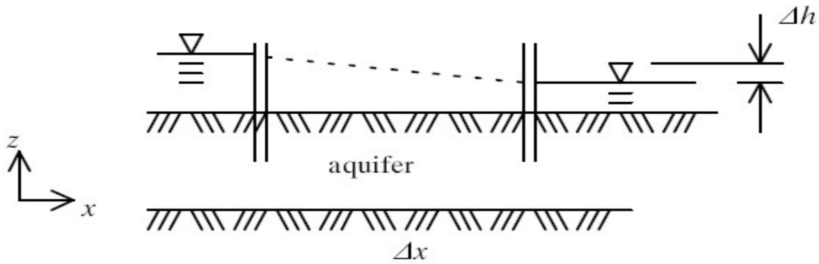


***Figure 2** A schematic diagram of groundwater situation in an area (e.g. a mining site)*



Numerical Examples

Although we have implemented the algorithm, and obtained the results, for different models, here only the results of a simple model, shown in Figure 2, have been presented. The boundary conditions $h_1 = 12\,\text{m}$ (left boundary) and $h_6 = 2\,\text{m}$ (right boundary) are given. Also we have considered the distance between the left and right boundaries to be $10\,\text{m}$ and the length of elements to be $2\,\text{m}$ (i.e. $\Delta x = 2\,\text{m}$). Thus, from the left to right, we have to obtain the hydraulic heads at nodes 2 to 5 (i.e. h_2 to h_5), having an equal distance interval $2\,\text{m}$ along x direction from each other. Also, we consider the hydraulic conductivity $K = 10^{-3}$ *m/s* (homogeneous medium) and the specific discharge at node 4, $q_3 = -10^{-3}\frac{1}{s}$ (sink), but no source or sink at other nodes, i.e. $q_i(i = 1, 2, 4, 5, 6) = 0\frac{1}{s}$. Using the finite difference solution algorithm, i.e. equation (10) and the procedure afterward, we solve the matrix equation $AH = b$, and thus, obtain: $h_2 = 7.6\,m$, $h_3 = 3.2\,m$, $h_4 = 2.8\,m$, and $h_5 = 2.4\,m$.

To obtain the flow (Darcy) velocity, we use equation (2), in which the first derivative $\frac{\partial h}{\partial x}$ can be obtained in three different ways, called forward differencing (FD), backward differencing (BD) and central differencing (CD). The FD and BD equations or methods are obtained from equation (7) and (8), respectively, after truncating the two expansions to the second term. The CD equation or method is also obtained by subtracting equation (8) from equation (7), after truncating the two expansions to the second term, i.e.

$$\text{FD:}\quad \frac{\partial h}{\partial x}=\frac{h_{i+1}-h_i}{\Delta x},\qquad \text{BD:}\quad \frac{\partial h}{\partial x}=\frac{h_i-h_{i-1}}{\Delta x},\qquad \text{CD:}\quad \frac{\partial h}{\partial x}=\frac{h_{i+1}-h_{i-1}}{2\Delta x}. \tag{12}$$

The CD method has a second (higher) order of accuracy, but it cannot be used for determining the flow velocity at the boundaries. Instead, the FD and BD methods, which have the first (lower) order of accuracy, are, respectively, used for the computation of the flow velocity at the left and right boundaries, as well as other (inner) nodes. Thus, we obtain the flow velocity quantities at the boundaries and the inner nodes using CD, FD and BD methods, as shown in Table 1.

Table 1 *The flow velocities at the boundaries and the inner nodes using CD, FD and BD methods*

velocity	CD method (*m/s*)	FD method (*m/s*)	BD method (*m/s*)
v_1 (left boundary)	-	$v_1=2.2\times10^{-3}$	-
v_2	$v_2=2.2\times10^{-3}$	$v_2=2.2\times10^{-3}$	$v_2=2.2\times10^{-3}$
v_3	$v_3=1.2\times10^{-3}$	$v_3=0.2\times10^{-3}$	$v_3=2.2\times10^{-3}$
v_4	$v_4=0.2\times10^{-3}$	$v_4=0.2\times10^{-3}$	$v_4=0.2\times10^{-3}$
v_5	$v_5=0.2\times10^{-3}$	$v_5=0.2\times10^{-3}$	$v_5=0.2\times10^{-3}$
v_6 (right boundary)	-	-	$v_6=0.2\times10^{-3}$

As can be seen from Table 1, the velocity at each of the inner nodes 2, 4 and 5, using all the three methods are the same, but the velocity at the inner node 3 is obtained differently using the methods.
The finite element modelling solution results for the same model were also obtained. Although the finite element results were more or less similar with the finite difference results, we could see a higher accuracy of the finite element results due to the basis of the method, especially using triangular element grid, caused that the irregular boundaries of the model were better resolved (in comparison to using rectangular meshes in the finite difference method). Also the finite element method was in practice more consistent and stable. Thus, finite difference method (in comparison to finite element method) is not a very accurate and proper method for modelling domains with complex geometries.

Conclusions

A finite difference modelling solution was presented for groundwater models. The models were discretised by rectangular element grids. The solution was applied to the models to obtain primarily unknown hydraulic heads, and then velocities, at the nodes of rectangular elements. Finally, the finite difference and finite element results for the same model were compared. The comparison showed that the finite element results were more accurate and the numerical procedure in this method was more consistent and stable. Due to using rectangular element grid in the finite difference method, complex boundaries and inner structures of the model can only be taken into account very roughly in this method, and thus, compared to finite element method, is not much suitable for modelling complex groundwater situations.

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