THE ELEMENT BALANCE METHOD OF HYDROGEOLOGIC COMPUTATION

Wu, Ho

ABSTRACT : The element balance method is one of the numerical methods in solving the problem of planed non-steady flow of groundwater. The vadose region can be divided into several elements by this method. And then, based on the water balance theory the equations of every nodes could be set up directly using a physical model. The solution of two-dimensional partial differential equation of groundwater flow can be described using a discrete equation group which contains only limited numbers of unknown variables. The application of this method could not only give hydrogeologic parameters, but also predict the draining for mine and estimate the groundwater resource. The paper introduced briefly this method.

RESUME : La méthode du bilan d'éléments est une des méthodes numériques pour résoudre le problème d'écoulement transitoire horizontal, des eaux souterraines. Selon cette méthode, on divise la région des gués en plusieurs éléments. Puis, basés sur la théorie du bilan d'eau, et employant un modèle physique, on établit directement les équations des divers noeuds. La solution de l'équation différentielle partielle dimensionnelle de l'écoulement de l'eau souterraine, peut être décrite par un groupe discret d'équations, qui seulement contiennent un nombre limité de variables inconnues. Ainsi, non seulement on peut obtenir des paramètres hydrogéologiques, mais aussi on peut prédire le drainage des mines et calculer les ressources en eau souterraine. Le travail ci-après présente brièvement cette méthode.

RESUMEN : El método de balance de elementos, es uno de los métodos numéricos para resolver el problema de flujo transitorio horizontal, de las aguas subterráneas. Según este método, se divide la región vadosa en varios elementos. Luego, basados en la teoría del balance del agua, y empleando un modelo físico, se establecen directamente las ecuaciones de los diversos nudos. La solución de la ecuación diferencial parcial bidimensional del flujo del agua subterránea, puede ser descrita por un grupo discreto de ecuaciones, que sólo contienen un número limitado de variables desconocidas. Así, no sólo se pueden obtener parámetros hidrogeológicos, sino también predecir el drenaje de minas y calcular los recursos subterráneos. El presente trabajo expone brévemente este método.

The Hydrogeologic Company of the Ministry of the Metallurgical Industry
The People's Republic of China, Peking, China.

1. Introduction

The use of finite element method in hydrogeologic computation began in the late of 1960's. In recent years the study of the usefulness of finite element method in hydrogeologic computation has been carried out by many hydrogeologists at home and abroad (Chang Hun-ren, Li Tzuhn-ting and international well-known scientists S.P. Neuman; etc). They discovered many disadvantages contained in the finite element method, because of symmetric matrices of the storage. Therefore, the discussion of the usage of finite difference method in hydrogeologic computation for irreqular grid became more and more fashion, but it is only limited in hydrogeologic computation of feeding water and the aquifer is considered isotropy. Refer to applying for the predict pit pumpage by these two methods of hydrogeologic computation in mining district, no papers have been published yet. The element balance method herein is based on the finite difference method for irregular grid and took in the reasonable form of finite nent method. Thus, it is either used for hydrogeologic computation of feeding water or predict the pit pumpage. Also, it takes into account ununiform medium of water-bearing bed and settles anisotropy of it.

2. Dividing the vadose region and establishing the model of water head.

Let's divide the vadose region and consider any plane element β and assume that its three point elements i, j, k, are anti-clockwise configuration (Fig 1). The value of surface h (x, y, t) of water head at the point element (L=i, j, k) is marked h_L (t). The coordinate of the point element L is (x_L, y_L). The linear interpolating function has to be done:

$$H^\beta(x, y, t) = ax + by + c$$

let

$$H^\beta(x_L, y_L, t) = h_L(t) \qquad L = i, j, k \qquad \cdots\cdots(1)$$

At the plane element β, the true function of water head can be approximated by the linear interpolation function H^β (x, y, t). The linear interpolating function H^β (x, y, t) is called the model of water head at β^{th} plane element.
From equation (1) we get:

$$H^\beta(x, y, t) = \sum_{L=i,j,k} \lambda_L h_L(t) \qquad \cdots\cdots(2)$$

Where:

$$\lambda_L = (\eta_L x - \xi_L y + \omega_L)/2\Delta_\beta, \quad L = i, j, k$$
$$\xi_i = x_j - x_k, \quad \xi_j = x_k - x_i, \quad \xi_k = x_i - x_j,$$
$$\eta_i = y_j - y_k, \quad \eta_j = y_k - y_i, \quad \eta_k = y_i - y_j,$$
$$\omega_i = x_j y_k - x_k y_j, \quad \omega_j = x_k y_i - x_i y_k, \quad \omega_k = x_i y_j - x_j y_i,$$
$$\Delta_\beta = (\omega_i + \omega_j + \omega_k)/2,$$

Δ_β is the area of the plane element β.

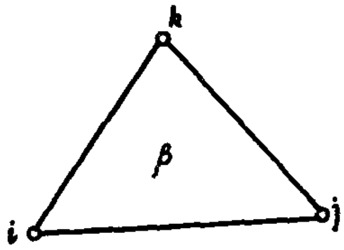

Fig 1

We assume the gradient of water head at direction x is $J_x^\beta(t)$ and one at direction y is $J_y^\beta(t)$, thus from equation (2) we obtain:

$$J_x^{\beta}(t) = \sum_{L=i,j,k} \eta_L h_L(t)/2\Delta_\beta$$

$$J_y^{\beta}(t) = \sum_{L=i,j,k} -\xi_L h_L(t)/2\Delta_\beta \qquad \cdots\cdots(3)$$

From this we can see that once we have determined the plane lements of vadose region, the model H^β (x, y, t) of water head at any plane element β will uniquely be determined by true water head h_L (t) (L=i, j, k) at its point element; the gradients along with directions x and y will be independent of variables x and y.

3. The establishment of the element balance equation

The establishment the element balance equation is closely correlated with the establishment of the balance element. The balance element R_i at an interior point element i will be set up in this way: the region R_i is consisted of connecting line between middle points appropriate line elements and the intersection of three perpendicular bisectors of the plane elements which have a common top the point element i (Fig 2). Actually, the establishment of the balance element is to subdivide the vadose region. After the subdividing, there must be no gap and superposition between the two neighboured balance elements. That is necessary condition which ensures that the approximate solution may converge to the precisional solution.

The mathematical expression of water quantity balance relation in the balance element is known as an element balance equation.

Now, let's deal with the main balance factors of a square iabc (a part of the plane element β) in the balance element R_i.

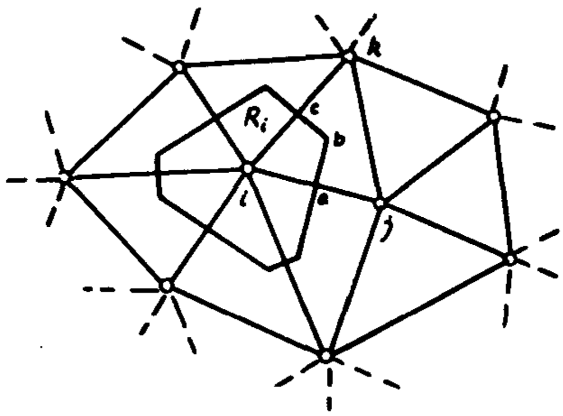

Fig 2

(1) Standing water storage or elastic water quantity

If we take $q_s^\beta(t)$ as the released standing water storage (for unconfined water) or elastic water quantity (for confined water) of the square iabc at time t, then it has:

$$q_s^\beta = \frac{S_\beta}{16\Delta_\beta} \mid (\eta_i\eta_k + \xi_i\xi_k)(\eta_j^2+\xi_j^2) + (\eta_i\eta_j + \xi_i\xi_j)(\eta_k^2+\xi_k^2) \mid \frac{\partial h_i(t)}{\partial t}$$

Where

$$S_\beta = \begin{cases} \mu^* M & \text{for the confined region} \\ \mu & \text{for the unconfined region} \end{cases}$$

M is the average thickness of confined aquifer, μ^* is confined water specific storage, μ is uncontined water specific yield.

The standing water storage or the elastic water quantity being released by R_i is:

$$Q_s(t) = \sum_{\beta=1}^{P} q_s^{\beta}(t) \qquad \text{.......................(4)}$$

Where the P is the numbers of all plane elements in the balance element R_i.

(2) Lateral flow

It is assumed that the lateral flow is positive when it is out from R_i, and negative when it runs into R_i (Fig 3). According to the Darsy theory, we take $q_u^{\beta}(t)$ as the lateral flow through the section abc, at a time t then it has:

$$\begin{aligned} q_u^{\beta}(t) &= \frac{x_k - x_j}{2} J_y^{\beta} T_y \\ &\quad - \frac{y_k - y_j}{2} J_x^{\beta} T_x \\ &= -\frac{\xi_i}{2} J_y^{\beta} T_y + \frac{\eta_i}{2} J_x^{\beta} T_x, \end{aligned}$$

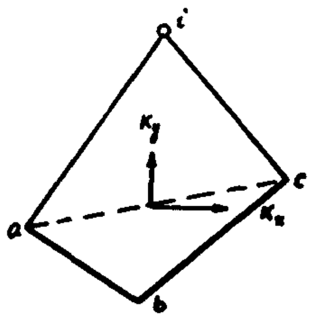

Fig 3

Substituting (3) in the above formula gives,

$$q_u^{\beta}(t) = \frac{\xi_i T_y}{4} \sum_{L=i,j,k} \xi_L h_L(t)/\Delta_\beta + \frac{\eta_i T_x}{4} \sum_{L=i,j,k} \eta_L h_L(t)/\Delta_\beta.$$

Where

$$T_x(t) = \begin{cases} K_x M & \text{in the confined region} \\ K_x \left[h_M^{\beta}(t) - B\right] & \text{in the unconfined region} \end{cases}$$

$$T_y(t) = \begin{cases} K_y M & \text{in the confined region} \\ K_y \left[h_M^{\beta}(t) - B\right] & \text{in the unconfined region} \end{cases}$$

B is average elevation of the base of plane element β, K is coefficient of permeability,

$$h_M^{\beta}(t) = \frac{h_i(t) + h_j(t) + h_k(t)}{3},$$

$Q_u(t)$ expresses the lateral flow of R_i, thus

$$Q_u(t) = \sum_{\beta=1}^{P} q_u^{\beta}(t) \qquad \text{.......................(5)}$$

For the balance element R_i, according to the continuity theory we have:

$$Q_s(t) - Q_u(t) = O \qquad \text{.......................(6)}$$

The equation (6) is called fundamental element balance equation of the balance element.

The equation (6) should be modified when the following yields take place on the plane element β in the balance element R_i

a. If there is a pnmping well with pumpage $q_w^{\beta}(t)$ in the plane element β (Fig 4), the $\frac{\Delta_i}{\Delta_\beta} \times q_w^{\beta}(t)$

should be placed the right side of the equal symbol, where Δ_i is the area coordinate of the pumping well.

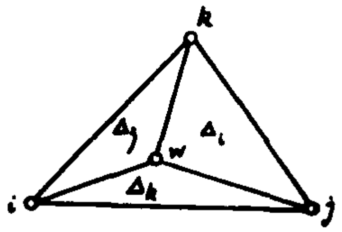

Fig 4

Let Q_w (t) be the total quantity of water given to R_i by the wells at all plane elements which have a common top point with the point element i, then the element balance equation is:

$$Q_s(t) - Q_u(t) = Q_w(t) \qquad \cdots\cdots(7)$$

b. If there is the infiltration water at balance element R_i and the infiltration in every plane elements is uniform. Let's use the symbol E_ρ to express the infiltration water quantity per unit area at per unit time, and the symbol $Q_E(t)$ to express the total infiltration quantity at the tlement R_i and we may have:

$$Q_E(t) = \sum_{\rho=1}^{P} \frac{E_\rho}{16\Delta_\rho} \mid (\eta_i \eta_k + \xi_i \xi_k)(\eta_j^2 + \xi_j^2) + (\eta_i \eta_j + \xi_i \xi_j)(\eta_k^2 + \xi_k^2) \mid ,$$

and the element balance equation will be:

$$Q_s(t) - Q_u(t) = Q_w(t) - Q_E(t) \qquad \cdots\cdots(8)$$

c. When i is the point element of second boundary, the balance element R_i is shown in Fig. 5. we assume that the flow of unit width section at second boundary is $q_B(t)$, and the outflow of R_i is positive, the inflow of R_i is negative, it should be written:

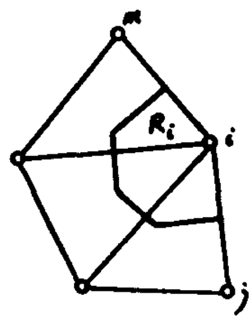

Fig 5

$$Q_E(t) = \frac{1}{2}(\overline{ij} + \overline{im})\, q_B(t)$$

Adding the above formula to the right side of equation (8) we have:

$$Q_s(t) - Q_w(t) = Q_w(t) - Q_E(t) + Q_B(t) \qquad \cdots\cdots(9)$$

The equation (9) is general form of the element balance equation.

When i is the point element of first boundary or the intersection between first and second bounderies we may not set up the equation. If there are N_1^{th} interior point elements and N_2^{th} point elements of second boundary, we may establish N^{th} ($N = N_1 + N_2$) element balance equations from equation (9) (Let $Q_B(t)$=O, when i is interior point element).

When the term $\frac{\partial h_i(t)}{\partial t}$ is obtained by the backward difference, the equation (9) will be the implicit equation. And then, all of the balance equations must be solved simulteneously. When it is obtained by the forward difference, the equation (9) will be the explicit equation. Therefore, the balance equations may be solved one by one. Because, various convergence rates may be produced in various balance elements for the same Δt, so in order to ensure the computational accuracy we should take the explicit equation in larger balance elements and the implicit equation in smallar balance elements. Thus, it may not only increase the computation speed, but also decrease the necessary memory capacity of a computer. In the unconfined water computation, we can properly take the average value of non-off quantities of two moments before and after the time difference for Q_u (t).

4. Inversion of parameters and predict of water level

Generally, the optimal value is choosed by optimization method 1 and primary linear programming within the range of every hydrogeologic parameters' exchanges. If a certain group of parameters can make the sum of absolute values of the differences between the observed values and computational values which are the water head of all measured wells and the moments be less then the accuracy index which is given before hand, this group of parameters will be regarded as hydrogeologic parameters.

In the case of given initial conditions, boundary condition, pumping intensity and hydrogeologic parameters, we can obtain the water head of each interior point elements and second kind of boundary point elements at moment $t_1, t_2, \ldots t_n$. The computational steps are as follows: first, compute the end-moment water head of first time interval. Second, take the result of the first step as the initial water head of second time interval to compute the end-moment water head of this interval. And then, use the same method sequently to compute the end-moment water head of the third, fourth . . . until the N^{th} time interval. We call this process water level predict. For paper limited specific steps about inversing the parameters and water level predict can not be given in detail.

In addition, the computation result is much dependent upon the boundary condition. In order to avoid subjectivity in general computational boundaries we must try our best to reach the natural boundaries.

Compared with the other numerical methods it has the advantages of clear physical concept and is easy for ordinary engineering technicians to master. Moreover, it may avoid disterbance of the local quality balance and the limitation of selected time intervals in the finite element method. We used this method in predicting the draining for mine and in estimating groundwater resource. As a rsult, we have received fairly good results.

REFERENCE

(1) Optimization method group of operation research office, Mathematical Institute of Chinese Academy. «Optimization Method», Science Publication House, 1975.