FREE SURFACE FLOW IN POROUS MEDIA BY FINITE ELEMENT METHODS

Martins, J. B.* (**) ; Matos, A. C.** ; Bianchi, A.***

ABSTRACT : The steady state flow through porous rigid media is analysed. Governing equations and boundary conditions are set up and discussed. Variational methods and finite element approach are described and the solving system of equations for quadrilateral networks of triangular linear condensed elements is obtained. Non-homogeneity and anisotropy are discussed. Further discussion on the finite element techniques and on the convergence of the numerical process. Some sugestions for improvement and quick determination of the exit point as a preliminary step to obtain the free surface are made, in order to reduce the total number of iterations. After the presentation of some examples, a rigorous formulation of the problem for heterogeneous media is put forward and since it cannot yet be practicized, the fundamental points for an approximate solution with a small number of iterations is reviewed.

RESUME : On analyse l'écoulement à travers des milieux poreux rigides. On établit et discute les équations qui gouvernent le phénomène et les conditions sur les frontières. On présente la méthode variationnelle avec approximation par des éléments finis et on obtient le système d'équations qui résoud le problème pour des réseaux quadrangulaires d'éléments triangulaires linéaires, condensés. On discute la non-homogénéité et l'anisotropie, les techniques des éléments finis et la convergence du processus numérique. Etant donnée l'importance de la position du point de sortie pour la fixation de la surface libre de l'écoulement, une suggestion est faite pour la détermination de ce point. Après la présentation de quelques exemples, on donne une rigoureuse formulation du problème pour les milieux hétérogènes et lorsqu'il n'est pas possible de la pratiquer, on avance les points à retenir pour une solution d'approche, avec un petit nombre d'itérations.

RESUMEN : Se analiza el flujo permanente, en medios porosos rígidos. Se establecen y discuten las ecuaciones básicas y las condiciones en los límites. Se describen los métodos variacionales y las aproximaciones por elementos finitos ; se obtiene la solución a los sistemas de ecuaciones para mallas cuadradas de elementos triangulares lineales condensados. Se discuten los problemas de la nohomogeneidad y anisotropía, de las técnicas de elementos finitosyde la conver-

* Universidade do Minho, Braga, Portugal
** Faculdade de Engenharia, Oporto, Portugal
*** Electricidade de Portugal (E. P.), Oporto, Portugal.

gencia del proceso numérico. Se hacen algunas sugerencias para mejorar la determinación rápida de la posición del punto de salida, como un paso para obtener la superficie piezométrica, y poder reducir el número total de iteraciones. Por último, tras mostrar algunos ejemplos, se presenta una formulación rigurosa para los medios heterogeneos, y, aunque no ha podido todavía ser practicada, se adelantan los puntos fundamentales para una solución aproximada con un número reducido de iteraciones.

1 - INTRODUCTION

The water percolation through porous media is governed by the well knwon Darcy's law. This experimental law states the proportionality between the specific flux vector $\underline{q}$ and the hydraulic gradiente $\underline{i}$.

$$\underline{q} = K\,\underline{i} = -K\,\mathrm{grad}\,\phi \tag{1}$$

The specific flux vector $\underline{q}$ is the flow rate per unit total surface through which flow takes place. Its components in cartesian x,y,z coordinates, are q_x, q_y, q_z respectively. The hydraulic gradient $\underline{i}$, is the piezometric head loss or energy loss due to friction per unit of length of percolation. Its components are $i_x = -\partial\phi/\partial x$, $i_y = -\partial\phi/\partial y$, $i_z = -\partial\phi/\partial z$.

When the medium is homogeneous and isotropic, in relation to percolation, K is a scalar constant, the permeability, and we may write (1) as:

$$q_x = K\,i_x = -K\frac{\partial\phi}{\partial x}\,;\quad q_y = K\,i_y = -K\frac{\partial\phi}{\partial y}\,;\quad q_z = K.i_z = -K\frac{\partial\phi}{\partial z} \tag{1'}$$

For the flow in the direction of the unit vector $\hat{s}$ we have

$$q_s = \underline{q}\cdot\hat{s} = -K\,\mathrm{grad}\,\phi\cdot\hat{s} = -K\frac{\partial\phi}{\partial s} \tag{2}$$

For homogeneous but, anysotropic media instead of a scalar K there exists a permeability matrix $[K]$ such that

$$\underline{q} = [K]\cdot\underline{i} = -[K]\cdot\mathrm{grad}\,\phi, \quad (3') \qquad K = \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix} \tag{3}$$

K is symmetric.

As a consequence of anysotropy the vectors $\underline{q}$ and $\underline{i}$ are noncolinear except in certain directions of space x_1, y_1, z_1, the principal directions or eigen vectors of the matrix $[K]$. This means that the directions of streamlines do not coincide with those of the normals to equipotentials (ϕ=constant).

Given the K_{ij} of matrix K the principal directions of permeability can be obtained by well know methods (e.g. Bear, 1972). If $\underline{1}, \underline{2}, \underline{3}$ are the principal directions of permeability and K_1, K_2, K_3 the eigen vectors of $[K]$ or principal permeabilities, the components of the specific flux vector q_x, q_y, q_z satisfy the equations

$$q_1 = K_1 i_1 \,,\quad q_2 = K_2 i_2 \,,\quad q_3 = K_3 i_3 \tag{5}$$

If the medium is non-homogeneous the permeability K would be a function of the space coordenates x,y,z. Hence instead of stating the relationship (1) we could state the following

$$\underline{q} = -\operatorname{grad}(K\phi) \tag{9}$$

where $K\phi = \varphi$ would be the velocity potential.

However is an erroneous form of stating Darcy's law (Bear, 1972), since in that case

$$\underline{q} = -(\operatorname{grad} K)\,\phi - \phi \operatorname{grad} K \tag{9'}$$

Therefore for ϕ =constant we could have flow due solely to variation of permeability which is impossible.

2 - GOVERNING EQUATIONS FOR STEADY FLOW

Mass conservation for a control volume imposes to the steady flow of a incompressible fluid through a rigid porous medium, the following relationship:

$$\operatorname{div} \underline{q} + \bar{Q} = 0 \,,\quad \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} + \bar{Q} = 0 \tag{10}$$

where $\bar{Q}$ is the externally applied flux, e.i., the volume of fluid externally added per unit of time and per unit volume of global flow space.

Since $\underline{q} = [K] \cdot \operatorname{grad} \phi$

We have

$$\operatorname{div}\left\{[K]\cdot \operatorname{grad}\phi\right\} + \bar{Q} = 0 \,,\quad \frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial \phi}{\partial x_j}\right) + \bar{Q} = 0 \,,\quad i,j = 1,2,3 \tag{11}$$

If x,y,z are the principal directions of permeability and $x_1=x$, $x_2=y$ e $x_3=z$, (11) becomes

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial \phi}{\partial z}\right) + \bar{Q} = 0 \tag{12}$$

For the solution of steady state or permanente flow problems we have to add to (11) or (12) boundary conditions.

3 - BOUNDARY CONDITIONS

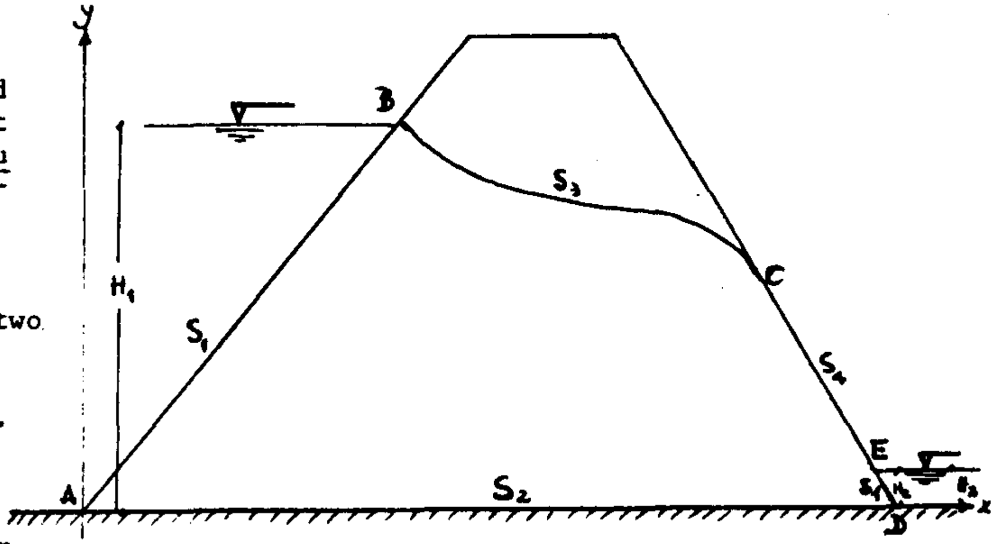
There are three kinds of boundaries: S_1 and S_4 (AB, ED and EC, Fig.1) where the potential is prescribed, ϕ =constant., S_2 (AB) where flux is prescribed and S_3 (BC) where the potential and the flux are prescribed.

A problem where there are conditions of potential type is called Dirichlet boundary value problem.

Conditions of prescribed flux refer to a diferent type of problem: the Neumann boundary value problem.

The problems as that of Fig. 1 where there are two or three types of boundaries is called mixed boundary value problems.

On surface BC (S_3) both potential and flux are prescribed. On the other hand the boundary itself is "apriori" unknown. The potential on the surface BC, called phreatic surface, must equalize the elevation had, i.e.,



$$\phi(x,y) = y \qquad (13)$$

and the normal component of the specific flux vector $\underline{q}$ must be null, i.e.:

$$q_n = \underline{q} \cdot \hat{n} = 0 \qquad (14)$$

where $\hat{n}$ is the unit outward normal to S_3.

In isotropic media the condition (14) becomes

$$\operatorname{grad} \phi \cdot \hat{n} = \frac{\partial \phi}{\partial n} = 0 \qquad (15)$$

On impervious boundaries S_2 (AD), the prescribed conditions are also (15) for isotropic media and (14) for anisotropic soils.

The value of potential on S_1 (AB) is ϕ=const=H_1 and on ED is $\phi = H_2$. On S_4 (seepage face CE), $\phi = y$.

Points such that C, E, D, B and A common to two kinds of boundaries are singular points. For correctness of the solution they must be treated in convenient way (Figs. 2 and 3).

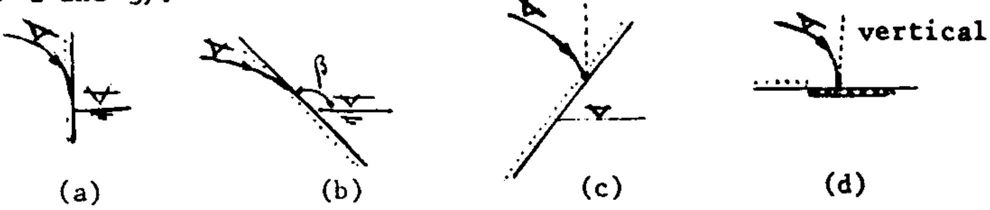


Fig. 2. Intersection of a phreatic surface with seepage face (isotropic media)

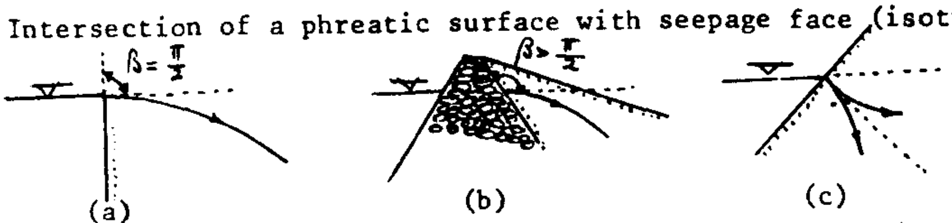


Fig. 3. Intersection of a phreatic surface with water table (upstream)

We notice that the phreatic surface is tangent to the vertical except when the merging angle $\beta > \frac{\pi}{2}$, in which case that surface is tangent to the seepage face (Fig. 2.b). In the case of the intersection with the up steam water table the phreatic surface is tangent to water table except in the case of $\beta < \frac{\pi}{2}$ (Fig. 3.c). For that case the phreatic line is tangent to the normal to the slope and may bend upwards (earth embankment) or downwards (ditch bank).

4 - ORDINARY METHODS OF SOLUTION

Analytical solutions on flow through porous saturated media without phreatic surface has been developed using the potential theory (Muskat, 1937, Polubarinova-Kochina, 1962). This type of approach is indicated for homogeneous aquifers and a wide range of shapes can be solved by maping techniques. However, for complicated geometry and boundary conditions as well as for non-homogeneous media and/or nonlinear flow, analytical methods are unsuitable.

Those difficulties have led to the development of numerical methods that permit the treatment of complex boundary conditions and a first approach to problems of heterogeneous anisotropic media.

Shaw and Southwell (1941) applied the relaxation method to the percolation through porous material and Finnemore and Perry (1968) adapted that technique to the use of computers. Other finite diference solutions have been obtained by Jeppson 1966, 1967, 1968 a, 1968 b, 1968 c, 1968 d, 1969 . Although finite diference techniques allow to deal with complex boundary conditions and phreatic surface problems as well as with anisotropy and heterogeneity the system of linear equations for solution in a computer cannot be easily set up. For these reasons Jeppson as limited his study to homogeneous media and have taken the cartesian coordinates as functions and potential ϕ and stream ψ as independente variables.

More recently finite elements approach has had wide spread application to field problems of all kinds due to easy fitting to boundary of complex geometry and to easy formulation and set up of the system of linear equations to which the problem is reduced.

Hundreds of papers, various treatises (see for ex. Zienkiewicz 1971, Desai 1972) have been writen on finite elements methods most of them relating its application to stress-strain problems in elastic media. Although an easy adaptation of elastic solutions can be done to fluid flow problems, a large number of papers have been writen also on finite elements applied to fluid mechanics. In particular free surface flow through porous saturated bodies (Taylor and Brown (1967); Finn (1967); Volker (1969); Zienkiewicz et al. (1966); Witherspoon et al. (1968); Neuman and Witherspoon (1970); France et al. (1971); Desai (1972 and 1976); Martins and Vargas (1976); Rui Correia (1977), etc.).

In what concerns phreatic surface problems a dificulty arises due to the fact that the numerical process must generate not only the values of the potential at the network points and rate of flow at points of known potential on the boundaries, but also the coordinates of the phreatic surface, itself. For this reason a trial phreatic surface is fixed "a priori" and changed in each iteration in order to fit both boundary conditions at S_3:

$$\phi(x,y) = y \quad , \quad (13) \quad \text{and} \quad \frac{\partial \phi}{\partial n} = c \tag{15}$$

Condition (15) is a noflow condition in the direction of the outward normal $\hat{n}$ i.e., the phreatic line is a stream line.

The changing of the trial phreatic surface can be acomplished either by moving the nodal points of the network (Taylor and Brown (1967); Neumann and Witherspoon (1970), R. Correia (1977)) along their columns or maintaining a fix network (Desai, 1976 a, 1976 b).

In the method of movable points there are two cases:.the first(Taylor and Brown 1967; Finn (1967)) assumes noflow on the starting phreatic line (condition (15)) and calculate the differences ϕ-y. After each iteration, the upper node in each column is changed in order to meet the condition (13). A new step follows, again assuming noflow at the new phreatic surface F.S..

In the second case of movable nodes (Neuman and Witterspoon (1970); R. Correia (1977)) each iteration is done in two steps: in the first step the condition (13) is imposed at the assumed phreatic surface and, in the second step geometry is not changed but the noflow condition (15) is imposed at the assumed phreatic surface and on the seepage surface (S_4, Fig. 1) the rate of flow calculated in the previous step is imposed at the nodal points. Further, the diferences ϕ-y are evaluated at the nodal points of the phreatic surface and the upper nodes changed in order to meet condition (13). Next a new iteration begins.

For the case of fixed network, Desai (1976), an initial F.S. is assumed, which may include all the domain inside the physical boundaries. A first solution is obtained assuming known potentials at S_1 and S_4 and considering nodes of S_3 as interior nodes which is equivalent to assume S_3 as a stream line.

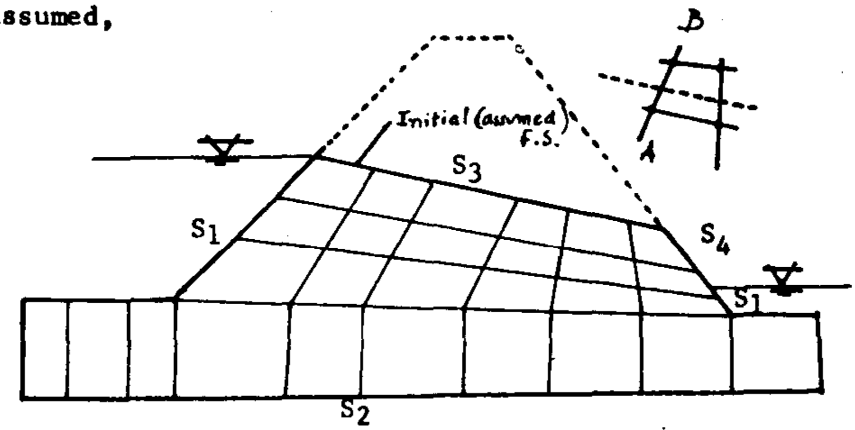


Fig. 3. Desai's fixed network

In a second step with the potentials knwon at nodes, points of ϕ=y are search for, along nodal lines such as AB. Usually, those will be found by interpolation between two consecutive nodes with potentials one less than y and the other greater than y. The curve joining such interpolated points gives the first approximation for the ES. Next, the elements through which the approximate ES passes are identified and the specific flux q_n normal to ES is calculated by means of the gradient of the known potential ϕ and the permability tensor. From the specific flows the flux is calculated at the neighbour nodes: the residual flow. Injecting at a further step fluxes at the same nodes but with changed sign a new set of potentials is obtained and from them a new position of the ES. The process is continued until ϕ_{i+1} do not differ "signifficantly" from ϕ_i.

5 - DISCUSSION OF THE METHODS OF SOLUTION.

Although free surface flow is a much discussed subject [Colin W. Cryer (Sept. 1977) elaborated a bibliography on that subject with 3.300 references], only recently Cryer and Fetter (June, 1977) nave proved the existence and uniqueness of solutions for the free surface problem. The same authors also proved the convergence of the numerical solutions based on the finite element methods to the exact solution. However, Neumann and Witherspoon relate dificulties of convergence near the exit node (Fig. 1) common to phreatic surface and downstream seepage face. Although those authors atribute that dificulty to Taylor and Browns method the fact is that they themselfs use a correction factor α and an additional correction β to get convergence in their method. Even so they admit that sometimes convergence is not reached after 25 iterations and provisions are taken in the computer program to stop calculation. Our experience with the method also confirms that for anysotropic media convergence may not be reached when the node C, where phreatic surface meets seepage face, is badly guessed at the initial step. R. Correia also refers the number of 32 iterations to get convergence by Taylor's method. In what concerns Desai's method the number of iterations is not published. However, Bromhead (1977) discussing Desai's paper (1976 a) call the atention to the singular entrance point B (Fig. 1). In that point, as we have seen (Fig. 3c) the phreatic surface is normal to the slope, but this condition as those refered in Fig. 2 for point C of merging between phreatic surface and seepage face are valid only for isotropic media. If the aquifer is anisotropic boundary conditions must be stated as follows:

$$q.\hat{n} = 0 \tag{16}$$

on impervious boundaries or on the phreatic surface. In the anisotropic case condition (16) is different from (15) since it involves the permeability tensor, i.e.,

$$\underline{q}\cdot\hat{n} = K_{ij}\cdot\frac{\partial\phi}{\partial x_i}\cdot n_j\ , \qquad i,j = x,y,z \tag{17}$$

On the other hand stream lines are no more normal to equipotential lines. In particular at the entrance point the phreatic surface is not normal to the upstream free. Since that angle is not constant but varies from node to node of the network, it is not always easy to fix "a priori" the angle between the phreatic surface and the upstream face at entrance point. Although condition (16) associated to condition (13) on neighbour points of the phreatic line will tend to fix the correct orientation of the phreatic surface at the entrance point, to ensure a perfect solution at that singular point as well as at the downstream exit point the best solution would be to map the actual flow region into another according to the ratio of the principal permeabilities $\lambda = K_1/K_2$. If the principal directions of permeability coincide with x and y axises the governing differential equation (12) becomes for homogeneous anisotropic media

$$K_x \frac{\partial^2 \phi}{\partial x^2} + K_y \frac{\partial^2 \phi}{\partial y^2} + K_z \frac{\partial^2 \phi}{\partial z^2} - \bar{Q} = 0 \qquad (18)$$

For two dimensional flow without acrestion $\bar{Q}$

$$\frac{\partial^2 \phi}{\partial \left(\frac{x}{\sqrt{\lambda}}\right)^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (19)$$

where $\sqrt{\lambda} = \sqrt{K_x/K_y}$. If we put $x' = x/\sqrt{\lambda}$ (21), i.e., if we divide the horizontal lengths by $\sqrt{\lambda}$, the anisotropic problem is reduced to an isotropic one. Also an heterogeneous ortotropic layered medium (Fig.16) can be transformed in a isotropic one.

In the case of anisotropy with the principal axies not parallel to the x and y the problem still can be reduced to an isotropic one (Harr, 1962).

The advantage of transforming the anisotropic medium in a isotropic one by simple changing the geometry before the numerical process is carried out, for free surface flow problems, is that the singular points where free surface flow meets upstream and downstream faces can be treated in a correct way at each iteration (Figs. 2 and 3). And such points are the sources of the instability of the numerical process (Neuman and Witherspoon).

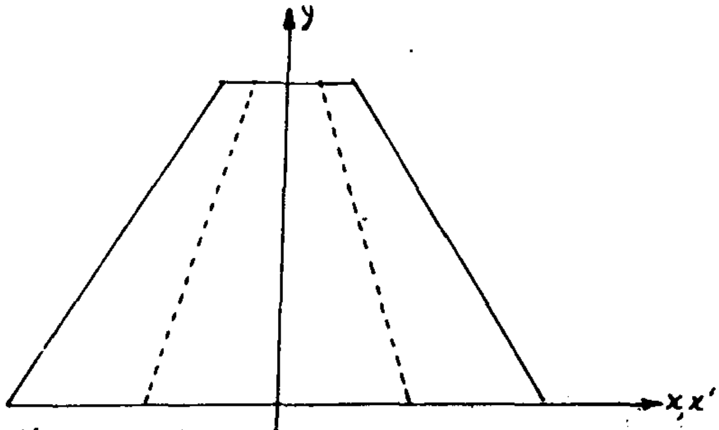


Fig.5 - Anisotropic medium contraction when $K_x > K_y$

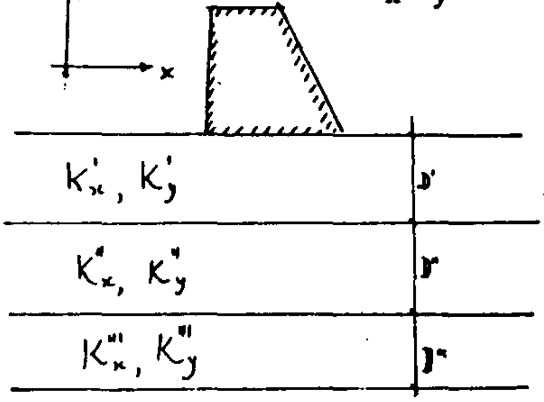


Fig.6 - Anisotropic heterogeneous layered medium

6 - FLOW THROUGH INHOMOGENEOUS MEDIA

For nonhomogeneous isotropic media Bear (1972) call the atention to the fact that $\Phi = K(x,y,z)\cdot\phi$ is not a velocity potential, as we have already seen (no.1) it was true hence we would have

$$\underline{q} = -grad(K\phi) = -(grad\,K(x,y,z))\cdot\phi - K(x,y,z)\,grad\,\phi$$

and for a constant hydraulic head ϕ we would have flow, due only to the variation of permeability, which is physically impossible. Therefore for continuously variable permeability we must write

$$\underline{q} = -K(x,y,z)\,grad\,\phi \qquad (3'')$$

However, the most common case is that in which there are two or more homogeneous media separated by surfaces where permeability is discontinuous. Such surfaces must be treated as internal boundaries satisfying certain conditions, e.g., the refraction law for the stream lines.

Let us consider a flow region divided in two subregions R' with permeability K' and R" with permeability K". We could solve the governing eq. (12) for ϕ, with $K = f(x,y,z)$ having a discontinuity over the curve C in the two dimensional case (Fig. 7). However, the best way is to divide the problem in two subproblems denoting potential in R' by ϕ' and in R" by ϕ''. We then search for a solution for ϕ' in R' and a solution for ϕ'' in R" satisfying their external boundary conditions on C' for ϕ' and C" for ϕ''. Also additional conditions must be stated on internal boundary C. Such conditions are:

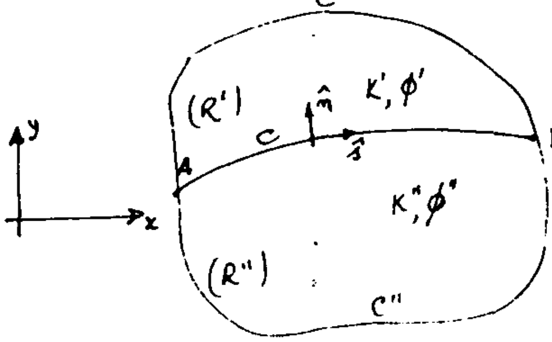


Fig. 7

$$\frac{\partial\phi'}{\partial s} = \frac{\partial\phi''}{\partial s} \quad \text{at all points of C} \qquad (20)$$

Also

$$\underline{q}'_n = \underline{q}''_n \;;\; \underline{q}'\cdot\hat{n} = \underline{q}''\cdot\hat{n} \;;\; (\underline{q}''-\underline{q}')\cdot\hat{n} = 0 \;;\; K'_{ij}\frac{\partial\phi'}{\partial x_i}\cdot n_j = K''_{ij}\frac{\partial\phi''}{\partial x_i}\cdot n_j \qquad (21)$$

From condition (20) we should have on C : $\phi''(s) = \phi'(s) + b$ (22)
where $\underline{b}$ would be an arbitrary constant. However, since on the points A and B common to both regions we must have $\phi'(A) = \phi''(A)$ and $\phi'(B) = \phi''(B)$. Hence (20) is equivalent to have

$$\phi'' = \phi' \qquad (23)$$

In the finite element discretization using isoparametric triangular elements (Neuman and Witherspoon (1970)) if C does not passes through any element, condition (23) is fulfilled on all points of C, since equalization of ϕ' and ϕ'' on any nodes $\underline{i}$ and $\underline{i+1}$ on boundary C implies equalization in every point of C between those nodes.

The same cannot be said from condition (21). This condition must substitute condition (11), valid for internal nodes. Therefore, the procedure of treating the inhomogeneous media as an homogeneous one with only different permeabilities from element to element when crossing the boundary C between two media, is incorrect and one should expect effects even on the convergence process.

7 - VARIATIONAL METHOD

As seen the continuity of flow gives the following governing equation for steady state:

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{y}\left(K_y \frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial \phi}{\partial z}\right) + \bar{Q} \tag{12}$$

with boundary conditions

$$\phi = \phi_o \text{ on } (AB \text{ and } CD \text{ in Fig.1}), \tag{24}$$

i.e., on the boundaries where potentials are fixed.

$$K_x \frac{\partial \phi}{\partial x} \cdot n_x + K_y \frac{\partial \phi}{\partial y} n_y + K_z \frac{\partial \phi}{\partial z} n_z + \bar{q} = 0 \tag{25}$$

on the boundaries where flux $\bar{q}$ per unit of surface is added or subtrated to the system.

For isotropic media ($K_x = K_y = K_z$) and on impervious parts of the boundaries (surface S_2 = AD on Fig. 1), (25) becomes:

$$\frac{\partial \phi}{\partial n} = 0 . \tag{25'}$$

For the phreatic surface S_3 there are two conditions:

$$\phi = \bar{y} \quad (13) \quad , \quad q_n = 0 . \quad (14)$$

(13) states that on S_3 the potential must be equal to elevation and (14) is the equivalente of (25) and becomes equal to (25') for isotropic media.

According the well-known Euler's theorem of the calcul of variations (12) is equivalent to the minimization of the funtional

$$w[\phi(x,y,z)] = \iiint_R \frac{1}{2} \left\{ \left[K_x \phi_x^2 + K_y \phi_y^2 + K_z \phi_z^2 \right] - \bar{Q}\phi \right\} dx\,dy\,dz \tag{26}$$

where $\phi_x = \frac{\partial \phi}{\partial x}$; $\phi_y = \frac{\partial \phi}{\partial y}$, $\phi_z = \frac{\partial \phi}{\partial z}$.

In fact Euler's theorem states that given the functional

$$w[\phi(x,y,z)] = \iiint_R f\left[x, y, z, \phi(x,y,z), \phi_x(x,y,z), \phi_y(x,y,z), \phi_z(x,y,z)\right] dx\,dy\,dz$$

the necessary and suficient condition to have a minimum over a bounded region R is that the unknown function f (x,y,z) satisfies the differential equation:

$$\frac{\partial f}{\partial \phi} - \frac{\partial f}{\partial \phi_x} - \frac{\partial f}{\partial \phi_y} - \frac{\partial f}{\partial \phi_z} = 0 . \tag{27}$$

Applying (27) to (26) we get (12).

If instead of the integral (26) the following is used

$$w(\phi) = \iiint_R \frac{1}{2} \left\{ \left[K_x \phi_x^2 + K_y \phi_y^2 + K_z \phi_z^2 \right] - \bar{Q}\phi \right\} dx\,dy\,dz + \iint_{S_2} \bar{q}\,\phi\, ds, \qquad (28)$$

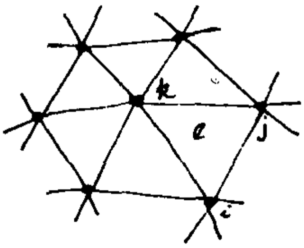
it can be shown (Zienckiewicz (1972)) that the minimization of (28) automatically includes the boundary condition (25).

8 - FINITE ELEMENTS APPROACH

In minimizing(28) by finite elements method the flow domain R is subdivided in a number of elements whose sides form a network.

Following the above author we substitute the unknown function $\phi(x,y,z)$ by a number of simple local functions N_i (x,y,z), N_j (x,y,z), N_k (x,y,z),..., defined on each element e,

$$\phi = \sum_e \left[N_i, N_j, N_k, \cdots \right] \{\phi\}^e, \qquad (29)$$



such that when $x=x_i$, $y=y_i$, $z=z_i$; $N_i = 1$ and $N_j = N_k = 0$

When $x = x_j$, $y = y_j$, $z = z_j$; $N_j = 1$ and $N_i = N_k = 0$

etc.. (the same for the other nodes of element e)

With this definition it is evident that the integral (28) becomes a summ over the number of elements in which the flow region have been subdivided and since N_i, N_j, N_k are known functions of x,y,z, the functional w will be converted in a function of the parameters ϕ_i, the values of ϕ at the nodes of the network. Therefore the problem of minimizing a functional w becomes a problem of minimizing a function of n variables $\phi_1, \phi_2, \ldots, \phi_n$ so many as the number of nodes where the potential is unknown.

To get the minimum of $w(\phi_i)$ we must have

$$\frac{\partial w}{\partial \phi_i} = 0 = \sum_e \frac{\partial w^e}{\partial \phi_i} \qquad (30)$$

where the sumation is extended to all the elements of region R and to boundaries, in what concerns the surface integral of (28).

For each element e

$$\frac{\partial w^e}{\partial \phi_i} = \iiint_{R_e} \left\{ K_x \phi_x \cdot \frac{\partial \phi_x}{\partial \phi_i} + K_y \phi_y \cdot \frac{\partial \phi_y}{\partial \phi_i} + K_z \phi_z \frac{\partial \phi_z}{\partial \phi_i} - \bar{Q} \frac{\partial \phi}{\partial \phi_i} \right\} dx\,dy\,dz + \iint_{S_2} \bar{q} \frac{\partial \phi}{\partial \phi_i} ds \qquad (31)$$

But

$$\phi_x \equiv \frac{\partial \phi}{\partial x} = \left[\frac{\partial N_i}{\partial x}, \frac{\partial N_j}{\partial x}, \frac{\partial N_k}{\partial x}, \ldots \right] \cdot \begin{Bmatrix} \phi_i \\ \phi_j \\ \phi_k \\ \vdots \end{Bmatrix}^e, \qquad (32)$$

etc.

etc. and therefore $\dfrac{\partial \phi_x}{\partial \phi_i} = \dfrac{\partial N_i}{\partial x}$, etc. (33)

Substituting (32) and (33) into (31), we have

$$\frac{\partial \omega^e}{\partial \phi_i} = \iiint_{R_e} \left(\left\{ K_x \left[\frac{\partial N_i}{\partial x}, \frac{\partial N_j}{\partial x}, \frac{\partial N_k}{\partial x}, \dots \right] \frac{\partial N_i}{\partial x} + K_y \left[\frac{\partial N_i}{\partial y}, \frac{\partial N_j}{\partial y}, \frac{\partial N_k}{\partial y}, \dots \right] \frac{\partial N_i}{\partial y} + K_z \left[\frac{\partial N_i}{\partial z}, \frac{\partial N_j}{\partial z}, \frac{\partial N_k}{\partial z}, \dots \right] \frac{\partial N_i}{\partial z} \right\} \left\{ \begin{matrix} \phi_i \\ \phi_j \\ \phi_k \\ \vdots \end{matrix} \right\}^e - \bar{Q} N_i \right) dx\,dy\,dz + \iint_{S_2^e} \bar{q} N_i \, dS = 0 \qquad (34)$$

Considering all the nodes of the element i,j,k,..., we may write in a compact form

$$[h]^e \cdot \{\phi\}^e + \{F\}^e = 0 \qquad (35)$$

where

$$h^e_{rs} = \iiint_{R_e} \left(K_x \frac{\partial N_r}{\partial x} \cdot \frac{\partial N_s}{\partial x} + K_y \frac{\partial N_r}{\partial y} \cdot \frac{\partial N_s}{\partial y} + K_z \frac{\partial N_r}{\partial z} \cdot \frac{\partial N_s}{\partial z} \right) dx\,dy\,dz \qquad (36)$$

is the "stifness matrix" and $r, s = 1, 2, \dots, N$ (N = nº of nodes)

$$\{F\}^e = -\iiint_{R_e} \bar{Q} \, N_r \, dx\,dy\,dz + \iint_{S_2^e} \bar{q} \, N_r \, dS \qquad (37)$$

is the "force vector".

Assembling the equations (35) for all the elements, we get the usual "equilibrium" equations:

$$[H] \cdot \{\phi\} + \{F\} = 0 \qquad (38)$$

where $H_{ii} = \sum_e h_{ii}$ (39) and $F_i = \sum_e F_i^e$ (39'), the summation beeing extended to all the elements that meet at node i, and $H_{ij} = \sum_e h^e_{ij}$ (39"), the summation being extended to the elements with the nodes i and j in common.

Before we go on, let us stress the analogy between (38) and the equilibrium equations for the elastic media. To the displacements in elastic body there corresponds the potentials in the flow medium. To the forces there corresponds rates of flow. Therefore $\bar{Q}$ is nothing but body forces and $\bar{q}$ is nothing but loads applied on the surface S_2. To impervious boundaries there corresponds surfaces of the elastic body without load and to equipotential boundaries there corresponds portions of the surface of the elastic body with imposed displacements. Certainly, the points on the boundary where displacements are imposed (null in particular) get reactions from the exterior. Therefore, points where potential is fixed have a rate of flow coming from or going to outside. If the body is in equilibrium as a whole, then the reactions must be in equilibrium with the surface loads, i.e., if there are no sources nor sinks, the rate of flow into the system at some boundaries must balance the rate of flow on the others, if the aquifer does not expand nor contract.

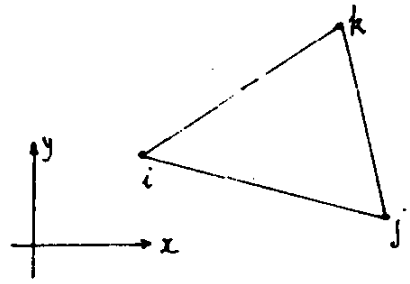
9 - SOLVING SYSTEM OF EQUATIONS FOR TRIANGULAR LINEAR ELEMENTS

If there is no flow in the z direction eqs.(36) and (37) become

$$h_{ij}^{e} = \iint_{R_e} \left(K_x \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + K_y \frac{\partial N_i}{\partial y} \cdot \frac{\partial N_j}{\partial y} \right) dx\, dy. \tag{40}$$

For the triangular linear element the shape function for node i is

$$N_i(x,y) = \left(a_i + b_i x + c_i y \right) / 2\Delta \tag{41}$$



where

$$a_i = x_j y_k - x_k y_j \;, \; b_i = y_j - y_k \;, \; c_i = x_k - x_j \tag{42}$$

$$2\Delta = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = 2 \cdot \left[\text{area of triangle } ijk \right] \tag{43}$$

As can be seen by substitution in (41), $N_i(x_i, y_i) = 1$

The other shape functions N_j and N_k would be obtained from (41) by circular permutation of subscripts.

Differentiating (41) and similarly for N_j and N_k and substituting in (41) one obtains the element "stiffness" matrix

$$[h]^e = \frac{K_x}{4\Delta} \begin{bmatrix} b_i b_i & b_i b_j & b_i b_k \\ & b_j b_j & b_j b_k \\ Sym & & b_k b_k \end{bmatrix} + \frac{K_y}{4\Delta} \begin{bmatrix} c_i c_i & c_i c_j & c_i c_k \\ & c_j c_j & c_j c_k \\ Sym & & c_k c_k \end{bmatrix} \tag{44}$$

Also performing the integration $\iint_{R_e} \bar{Q} N_i \, dx\, dy = \iint \frac{\bar{Q}}{2\Delta} \left(a_i + b_i x + c_i y \right) dx\, dy$

the body force vector $\left\{ f \right\}_e^e = - \frac{\bar{Q}\Delta}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$ (45)

would be obtained; i.e. the rate of flow per unit volume $\bar{Q}$, assumed to be constant, can be divided in equal parts by the three nodes of the element.

Having scattered the h_{ij}^{e} of each element on the nodes of the network as refered in (39), (39') and (39"), we come to the system of linear equations

$$H_{rs} \phi_s - F_r = 0 \qquad r, s = 1, 2, \cdots, N \tag{46}$$

where N is the number of nodes.

Some of the nodes, say M, lie on the boundaries S_1, S_3 and S_4 (Fig. 1) where the potential is stated. Therefore the number of unknowns can be reduced to N-M. After putting in right hand side of (46) the known values, we have

$$H_{\lambda n}\phi_n = F_\lambda - H_{\lambda m}\phi_m \ , \qquad m \neq n \tag{47}$$

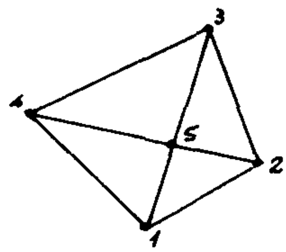
where $\underline{m}$ are the nodes where the potential is fixed.

After getting the N-M potentials from the linear system of equations (47), we can substitute them in the remaining set of M equations.

$$H_{mn}\phi_n = F_m \tag{48}$$

and obtain the rates of flow at nodes $\underline{m}$.

From a computer programming point of view it should be notice that the partial matrices H_{rn} of the coefficients and H_{rm} of the constant terms can easily be extracted from the global "stiffness" matrix H_{rs}; r,s = 1,2,...,N, if we codify the nodes where the potential is unknown, say nodes type 0, and the nodes where the potential is known, say nodes type 1.
It should also be refered that although the basic "stiffness" matrix is that of the triangular element, the "stiffness" matrix for a quadrilateral element is obtained by condensation at a further step in the program (Desai and Abel, 1972).



10 - NON-HOMOGENEITY AND ANISOTROPY

Zienkiewicz (1971) call the attention to the fact that the functional (28), to be minimized, has no derivatives of the permeability K_x, K_y and K_z. From that situation he infers that the permeabilities can change abruptly between elements or be allowed to vary within, since account of such a variation is taken in integrals evaluating the element matrices. However, one should notice that, even in homogeneous media there is no continuity of specific flux rate q_n normal to the common side of two contiguous elements, as there should be. This happens, for example, in the triangular linear elements, which have continuity of ϕ on the sides in common, but not continuity in the derivatives of ϕ and therefore of q_n. Of course as long as the number of elements tends to infinity the "chapeau" functions, by which we substitute the potential, tends to the actual $\phi(x,y)$ and therefore the derivatives of ϕ tend to be continuous and hence q_n becomes continuous at the limit. In the case of non-homogeneous media we already have seen (21) that the continuity of q_n must be explicitment formulated at the internal boundary C (Fig. 7) where permeability K_{ij} is discontinuous. The increasing number of elements will assure continuity of the derivatives of ϕ but not that of $q_n = \underline{q}\cdot\hat{n} = K_{ij}\dfrac{\partial\phi}{\partial x_i}n_j$.

Therefore for the case of abrupt change on an internal boundary C (Fig. a) (21) and (23) should be used instead of the "equilibrium" equations (38). Since for triangular linear elements $\dfrac{\partial\phi}{\partial x} = \text{const} = b_i\phi_i/2\Delta$ and $\dfrac{\partial\phi}{\partial y} = \text{const} = c_i\phi_i/2\Delta$ (49)

(21) would simply become

$$\begin{bmatrix} K'_{xx} & K'_{xy} \\ K'_{yx} & K'_{yy}\end{bmatrix} \begin{Bmatrix} \sum b'_i \phi'_i \\ \sum c'_i \phi'_i \end{Bmatrix} \frac{1}{2\Delta'} [n_x, n_y] =$$

$$= \begin{bmatrix} K''_{xx} & K''_{xy} \\ K''_{yx} & K''_{yy}\end{bmatrix} \begin{Bmatrix} b''_i \phi''_i \\ c''_i \phi''_i\end{Bmatrix} \frac{1}{2\Delta''}[n_x, n_y] \quad (50)$$

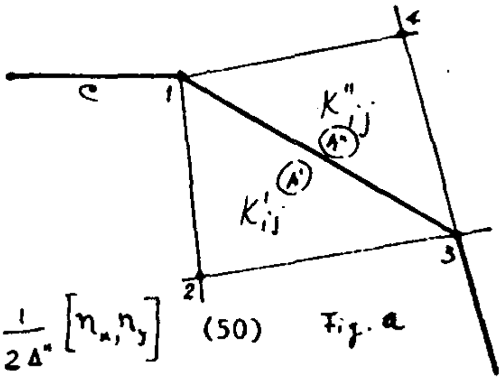


Fig. a

with $b'_i = y_i - y_k$, $c'_i = x_k - x_j$, etc.,

where i,j,k are circular permutations of 1,2,3 and 1,3,4 for triangles A' and A" respectively.

In the case of homogeneity but with anisotropy the element "stiffness" matrix must be calculated in local coordinates x_1, y_1 i.e. in the principal directions of the permeability. However, since potentials ϕ are scalar quantities, the assembling of the "stiffness" matrix for the whole network can be perfomed in the usual way.

Before we go further let us have a brief reference in relation to the best type of elements to be used. The linear triangular element used by Zienckiewicz (1971), Neuman and Witherspoon (1970) and others, has the advantage of simplicity and the values of ϕ are continuous on the side common to each pair of contiguous elements. As refered above (No.9) the triangular mesh can be transformed in a quadrilateral one, by condensation, but that does not increase accuracy. Recently other types of elements have been used such as the quadrilateral quadratic element (R. Oliveira (1977)). Although a higher degree accuracy for this type of element may be espected, continuity of ϕ and its derivatives on the common side of each pair of contiguous elements remains to be shown.

11 - FURTHER DISCUSSION OF FINITE ELEMENTS TECHNIQUE AND SOME SUGESTIONS FOR IMPROVEMENT

We have already seen (No.5) that methods based on deformation of the upper elements may do not converge near the exit point where phreatic surface meets the seepage face (point C in Fig. 1). Neumann and Witherspoon (1971) also refer outflow at the singular point common to the upstream face of the dam and the impervious base. Also the number of iterations necessary to get convergence is rather large. We think that this large number of iterations and also the instability of the process has something to do with the relationship between the imposed maximum error $|y-\phi| < \varepsilon$ and the mesh size. In fact with a large mesh size we can expect that the difference $y-\phi$ changes sign from one iteration to the next maintaining in both an absolute value $|y-\phi|$ greater than the imposed maximum error ϵ.

Another point to refer is the error in the rate of flow. Neumann and Widerspoon (1971) and others impose no maximun error on the rate of flow at free surface. However, when the condition $|y-\phi|<\epsilon$ is reached the flow through the final phreatic surface will not be zero. Our experience have shown that in some cases the minimum $|y-\phi|$ in the process does not correspond to the minimum flow through the phreatic surface.

Since a good approximate initial position of the exit point C is essential for "convergence" in any method, we must state a way of determinating that position as a preliminary phase. Going into the physical process we can see that with a few initial steps we can obtain pratically the final position of the exit point C.

In fact if a trial free surface (assumed a polygonal or straight line) is fixed at a high level and we impose $\phi = y$ on AC_1, the system will respond with an average inflow through AC_1. If AC_n is fixed too low, and $\phi = y$ - the system will respond with an average outflow through AC_n. Therefore, the correct exit point C is that between those sucessive points C_i an C_{i+1} for which the average rate of low through AC_i and AC_{i+1} are of oposite signe. Therefore the exit point may be obtained interpolating between hese residual average rates of flow of oposites signs of the last steps. We get the position of C and although that of a provisional free surface. This is obtained by interpolation between AC_{n-1} and AC_n.

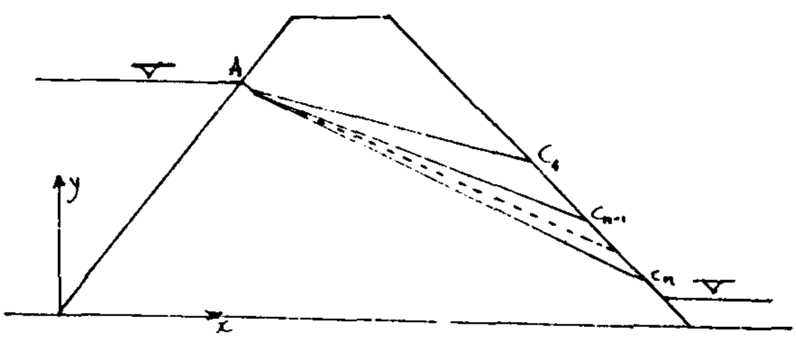


Fig. 8

After this preliminary steps the change in the shape of the free surface, assumed at start, can be done by Desai's fixed network, for example. If in a different way we take AC_n too low and assume no flow on it, the system will respond with potential ϕ_{C_n} larger than y_{C_n} since the flow region is restricted. Then we must move C uppwards until $\phi_{C_n} - y_{C_n} < 0$.

12. SOME EXPERIENCES USING SOME OF THE SUGESTIONS MADE

As a first example we refer an homogeneous isotropic dam (Fig. 9) with a vertical filter. We started with a trial exit point C_1 at the elevation of 84,00 m and went down to elevations of 74.00 m, 64.50 m and 54.00 m, imposing in each calculation $\phi = y$ on AC_i. For elevation of 64.50 we obtained the average inflow of $+.46\times10^{-6} m^3/s.m$ through AC_3 and for elevation of 54.00 we obtained an outflow of $-.32\times10^{-6} m^3/s.m$ through AC_4. Doing a linear interpolation between these positions of C in proportion of the flow rates for the intermediate points, we obtained a tentative free line which practically coincides with that obtained by a large number of iterations with a deformable mesh (Martins and Vargas, 1976).

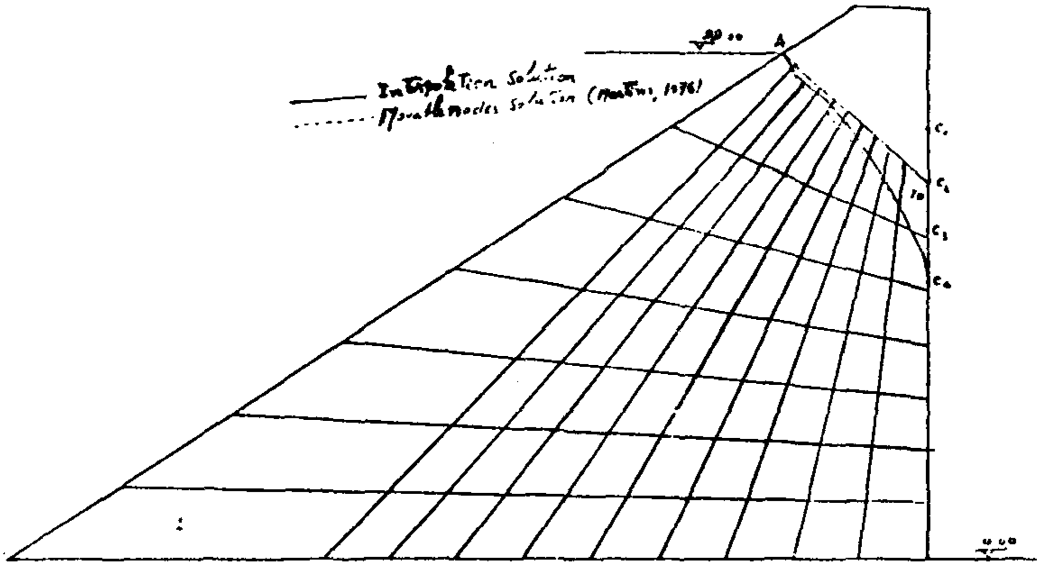


Fig. 9 (1st. Ex.) - Homogeneous dam with vertical filter. F.E. mesh.

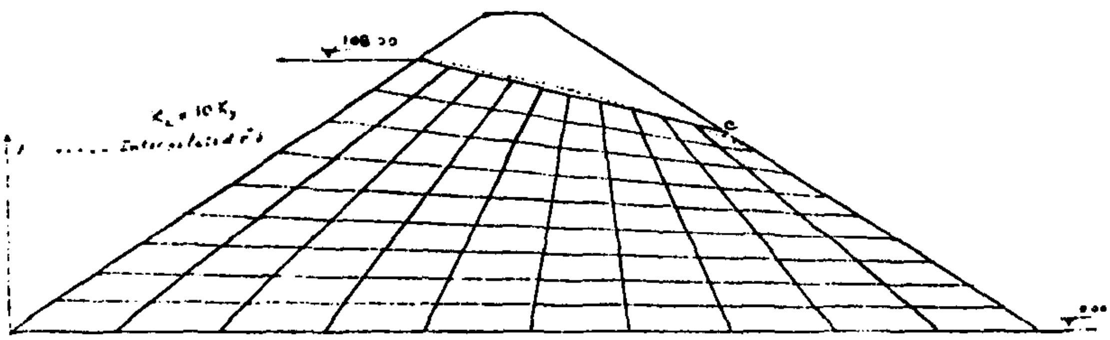


Fig. 10 (2nd. Ex.) - Anisotropic dam F.E.mesh. Motion on F.S. after determination of exit point C.

A second example refers to an anisotropic homogeneous dam (Fig.10). The ratio between the horizontal and vertical permeabilities is 10.We first got an approximate exit point at elevation 80.00. Further we impose the condition of noflow on the estimated free surface and did the interpolation sugested by Desai (1976). The result very much approaches a preceedings one with a large number of iterations (Martins and Vargas, 1976).

As a third example we present the seepage flow out of a ditch . Although the first interpolation according to Desai seems not to give a good free surface line (Fig. 11), nevertheless the flow rate out of the ditche pratically coincides with that given by Jeppson (1968 c).In fact we obtained the total outflow of $Q_t = 0{,}995 \times 2m^3/(s \times m)$ for $K = 10^{-6}$ m/s which gives $Q_t/KD = 6{,}6$ pratically coincident with that given by the same author (1968 a, p.280, T/D = 4 and H/D = 4).

As a forth example (Fig. 12) the problem of the seepage through an heterogeneous aquifer is solved by interpolation within a fixed mesh with no flow at the upper surface (Desai, 1976) and,alternatively, solved after the exit point is obtained by trial according the foregoing sugestion. No significative diferences have been found between the two techniques. In both cases the rate of flow is about 20% higher than that given by the Dupuit approximation (Bear, 1972, p. 373).

Finally, a fifth example (Fig.13) deals with an heterogeneous, anisotropic earth dam. The problem have been solved with a fixed network. The first interpolation according to Desai gives a solution somewhat far from Correia's one (1977). It should be notice that both methods does not includ discontinuity of permeability on the boundaries between the two media as an autonomous internal boundary (no. 6). Correia's program uses quadrilateral quadratic elements and ours uses a simple quadrilateral mesh obtained by condensation of triangular elements. His solution has a much smaller drop of potential through nucleous than ours.

13 - LAST DISCUSSION AND CONCLUSIONS

The convergence of the finite elements approach to the solution of the free surface flow problem means essentially that for properly posed problems when the maximum size of the mesh tends to zero the finite elements solution tends to the exact one, which is unique. It does not mean that for a given mesh with a fixed number of nodal points and a given form of discretization there is a polygonal free surface such that the potential ϕ exactly coincides with the elevation y, the rate of flow being simultaneously null at the vertices.

Therefore, for a given mesh, we cannot impose at will a limit to the maximum error $\varepsilon = |\phi - \bar{y}|$, independent of the mesh size. However, we feel that for a given mesh and a given form of discretization, there is a polygonal free surface which minimizes the error ε. To get it we might begin to consider a generalized form of funtional $\omega = \int_V f[x, y, \phi(x,y), \phi_x, \phi_y, \bar{y}(x,y)]\, dv$ associated to the functional $\varepsilon = |\phi - \bar{y}|$, both to be simultaneously minimized within the physical domain where the flow can exist . The solution of such a problem would require basic theoretical knowledgements perhaps not yet available.

Meanwhile we may search the best position of the free surface according the following lines:

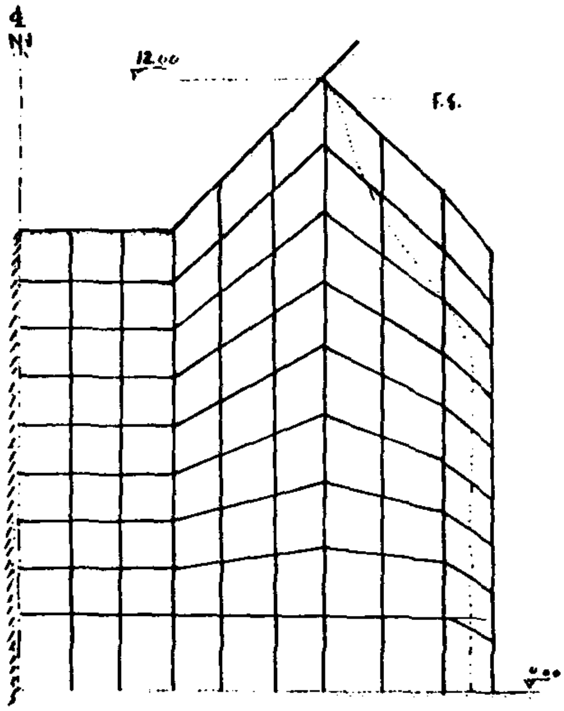


Fig. 11 (3rd. Ex.) - Flow out of a ditch. F.E.mesh Desai's first interpolacion for F.S.

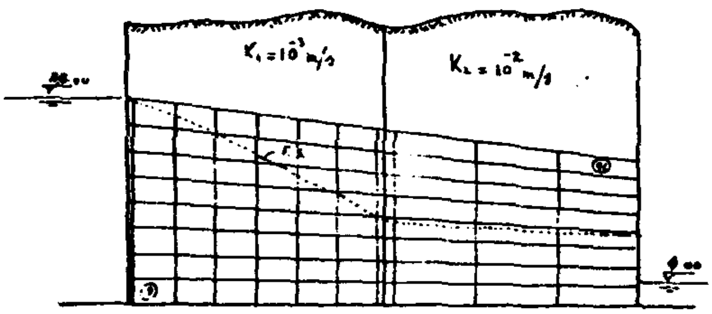


Fig. 12 (4th. Ex.) - Seepage through heterogeneous aquifer. F.E.mesh.

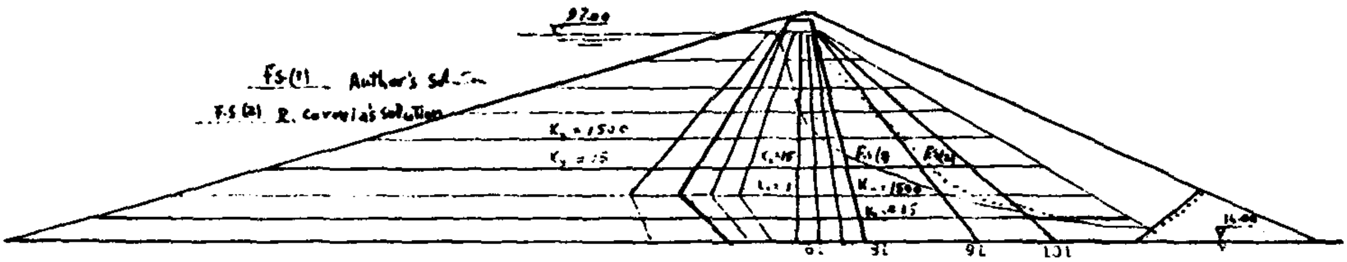
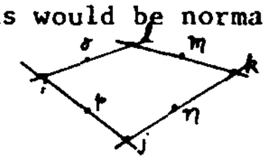


Fig. 13 (5th Ex.) - Anisotropic heterogeneous dam. F.E.mesh.

1. Begin with the determination of exit point since this point is critical in relation to the position of the free surface as stated before (no.11) at the provisional free surface assumed to be some polygonal (e.g. a straight line) between the entrance point and the current tentative exit point. Calculating the flow rate Q_i at each nodal point $\underline{i}$ at on the corresponding free surface, we average Q_i. When Qaverage becomes negative, i.e., the flow domain becomes so restricted that on the trial free surface there is outflow, as an average, we stop the process and obtain the approximate exit point by interpolation between the two last positions. We also get an interpolated free surface. Further, we recalculate - setting a noflow condition on the just obtained free surface. The actua shape of the free surface may now be obtained doing the kind of interpolation sugested by Desai (1976) for a fixed network.

ii. For a properly posed problem in heterogeneous media, we must consider internal boundaries where the permeability is discontinuous. This cannot be done correctly treating the nodes at those boundaries as internal normal nodal points, unless we used a kind of mixed finite element where at corners i,j,k,l the unknows would be potentials and at midsides m,n,o,p the unknowns would be normal specific rates of flow q_n.



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