COMPLEX CONTINUUM MODEL FOR DESCRIPTION OF THE SIMULTANEOUS SOLID-FLUID MOVEMENTS

Richter, R. and Bobok, E.

ABSTRACT : The purpose of this paper is to set out the fundamental concepts, definitions and laws of continuum mechanics applying for non-homogenous systems of rock, water and gas. In this treatise all scalar, vector and tensor fields are multiplied, as multiplied the number or the phases. On the basis of these simultaneous field concept the equations of change for mass, momentum and energy are presented.

RESUME : L'objectif de cet article est d'établir les fondements du mode d'exposé des mouvements simultanés des systèmes non homogènes de composants continus des solides et des liquides. Dans notre description, tous les champs scalaires, vectoriels et tensoriels se multiplient en fonction des nombres de phases. Nous avons introduit les équations de compatibilité, les équations des mouvements et les équations de l'énergie sur la base de la théorie des fonctions spatiales.

RESUMEN : El objetivo de esta comunicación es establecer los conceptos fundamentales, definiciones y leyes, relativos a los movimientos simultáneos en sistemas no homogeneos de roca, agua y gas. En esta descripción, todas las magnitudes, ya sean escalares, vectoriales o tensoriales, quedan multiplicadas por el número de fases. Sobre la base de la teoría de funciones espaciales hemos introducido ecuaciones de compatibilidad, del movimiento, y de transferencia de masas y energía.

Dr. Ing. Richard Richter, Professor of Geomechanics, University for Heavy Industry
3515 Miskolc, Egyetemváros - Hungary
Dr. Ing. Elemér Bobok, Senior Research Engineer, University for Heavy Industry
3515 Miskolc, Egyetemváros.- Hungary

The need to investigate the mechanical properties of rock has especially stimulated research in a field where interactions occur between the solid rock and a fluid phase: water or gas. This paper sets out a new mathematical foundation of the mechanical behaviour of rock with water or gas in its pores. It does not attempt to give direct solutions for mining or engineering problems. We wish this paper could be a preliminary condition for the later applications.

The base of this work is the apparatus of continuum mechanics. Our fundamental assumption is that the multicomponent solid-fluid system can be considered as a complex continuum. The detailed fine structure of the rock and the fluid in its pores can be replaced by a continuous model of matter having to some extent continuum properties. The essential mathematical idea of continuum model is that the real properties of matter are substituted by continuously distributed field functions of these properties for exemple: density, stress, strain, displacement, gravity force etc. These physical fields are multiplied simultaneously for multicomponent mechanical systems as they follow:

$$\begin{array}{lll}
\varrho_S = \varrho_S\,/\vec{r},t/ & \varrho_F = \varrho_F\,/\vec{r},t/ & \varrho_G = \varrho_G\,/\vec{r},t/ \\
\vec{v}_S = \vec{v}_S\,/\vec{r},t/ & \vec{v}_F = \vec{v}_F\,/\vec{r},t/ & \vec{v}_G = \vec{v}_G\,/\vec{r},t/ \\
\underline{\underline{F}}_S = \underline{\underline{F}}_S\,/\vec{r},t/ & \underline{\underline{F}}_F = \underline{\underline{F}}_F\,/\vec{r},t/ & \underline{\underline{F}}_G = \underline{\underline{F}}_G\,/\vec{r},t/
\end{array}$$

The indices S, F and G refer to the solid, liquid and the gas phases. Like the density, velocity and stress similar multiplication is obtained for other scalar, vector and tensor fields.

Thus the material derivatives are also multiplied:

$$\frac{d..}{dt_S} = \frac{\partial ..}{\partial t} + /\vec{v}_S \nabla / \ldots$$

$$\frac{d..}{dt_F} = \frac{\partial ..}{\partial t} + /\vec{v}_F \nabla / \ldots$$

$$\frac{d}{dt_G} = \frac{\partial ..}{\partial t} + /\vec{v}_G \nabla / \ldots$$

Assuming that the pores can be ignored individually it is convenient to imagine the infinitesimal volume of the complex continuum containing many pores to interpret a scalar field of porosity by definition

$$\Phi /\vec{r},t/ = \frac{dV_{pore}}{dV} .$$

If $\Phi /\vec{r},t/$ is a continuous field and the pores are filled perfectly with liquid and gas, we get for the infinitesimal volume

$$dV = dV_S + dV_F + dV_G = /1 - \Phi / \; dV + \Phi \; dV$$

Let the amount of the pore volume filled with fluid α_F and with gas α_G. Thus we get the following expression:

$$dV = /1 - \Phi / \; dV + \alpha_F \Phi \; dV + \alpha_G \Phi \; dV$$

Naturally:

$$\alpha_F + \alpha_G = 1.$$

Let V = V/t/ denote an arbitrary material volume of rock with fluid and gas in its pores. The mass of the matter in this material volume does not change as V moves. This is the principle of conservation of mass, which can be express by the equation:

$$\frac{d}{dt}\int_{V/t/}\left[\,/1-\Phi/\varrho_S+\alpha_F\Phi\varrho_F+\alpha_G\Phi\varrho_G\right]dV=0.$$

The substantial time derivative of the total amount of mass will be separated to local and convective terms

$$\int_{V}\frac{\partial}{\partial t}\left[\,/1-\Phi/\varrho_S+\alpha_F\Phi\varrho_F+\alpha_G\Phi\varrho_G\right]dV\;+$$

$$+\int_{/A/}\left[\,/1-\Phi/\varrho_S\vec{v}_S+\alpha_F\Phi\varrho_F\vec{v}_F+\alpha_G\Phi\varrho_G\vec{v}_G\right]d\vec{A}=0,$$

in which the first integral is taken through the control volume V and the second integral is taken over the closed surface /A/, bounding V. The first integral represents a time rate of accumulation of mass within the control volume, the second is the algebraic excess of the outflux of mass through the control surface over the influx of mass. The principle of conservation of mass is valid for both the rock, the fluid and the gas separately:

$$\int_{V}\frac{\partial}{\partial t}\left[\,/1-\Phi/\varrho_S\right]dV+\int_{/A/}/1-\Phi/\varrho_S\vec{v}_S\,d\vec{A}=0$$

$$\int\limits_V \frac{\partial}{\partial t} /\alpha_F \Phi \varrho_F/ \, dV + \int\limits_{/A/} \alpha_F \Phi \varrho_F \vec{v}_F \, d\vec{A} = 0$$

$$\int\limits_V \frac{\partial}{\partial t} /\alpha_G \Phi \varrho_G/ \, dV + \int\limits_{/A/} \alpha_G \Phi \varrho_G \vec{v}_G \, d\vec{A} = 0$$

This scalar integral equations can be replaced by their differential forms, since V is arbitrary

$$\frac{\partial}{\partial t} \left[/1 - \Phi / \varrho_S \right] + \operatorname{div} \left[/1 - \Phi / \varrho_S \vec{v}_S \right. = 0$$

$$\frac{\partial}{\partial t} /\alpha_F \Phi \varrho_F/ + \operatorname{div} /\alpha_F \Phi \varrho_F \vec{v}_F/ = 0$$

$$\frac{\partial}{\partial t} /\alpha_G \Phi \varrho_G/ + \operatorname{div} /\alpha_G \Phi \varrho_G \vec{v}_G/ = 0$$

The equations are valid ignoring the interphase mass transport. Otherwise source terms must be occured on the right side of the equations. If $\Phi = 0$ we get the continuity equation for homogeneous rock. If $\Phi = 1$ and $\alpha_F = 1$ the second equation turns to the continuity equation for pure fluids. Finally the $\Phi = 1$, $\alpha_G = 1$ case refers to pure gases.

We consider now the dynamics of the motion. The fundamental principle of the dynamics the law of conservation of momentum. As it is known this statement means that the rate of change of momentum of a material volume V equals the resultant force on the volume.

The momentum of the infinitesimal volume of the complex continuum is

$$[/1 - \Phi/\varrho_S \vec{v}_S + \alpha_F \Phi \varrho_F \vec{v}_F + \alpha_G \Phi \varrho_G \vec{v}_G]\, dV .$$

Its rate of change expanding to the whole volume V is

$$\frac{d}{dt} \int_V [/1 - \Phi/\varrho_S \vec{v}_S + \alpha_F \Phi \varrho_F \vec{v}_F + \alpha_G \Phi \varrho_G \vec{v}_G]\, dV =$$

$$= \int_V [/1 - \Phi/\varrho_S \vec{g}_S + \alpha_F \Phi \varrho_F \vec{g}_F + \alpha_G \Phi \varrho_G \vec{g}_G]\, dV +$$

$$+ \int_{/A/} [/1 - \Phi/\, \underline{\underline{F}}_S + \alpha_F \Phi\, \underline{\underline{F}}_F + \alpha_G \Phi\, \underline{\underline{F}}_G]\, d\vec{A} .$$

Here $\vec{g}$ is the extraneous force per unit mass. We tacitly assume that the force $\vec{g}$ is a known function of position and time. We also adopt the stress principle of Cauchy, which states that upon any surface exist a distribution of the surface forces depending at any given time on the position and the orientation of the surface. This relation between the stress vector $\vec{f}$ and the direction of the surface element can be expressed by the stress tensor $\underline{\underline{F}}$ as

$$\vec{f} = \underline{\underline{F}}\, d\vec{A} .$$

Applying Gauss' divergence theorem, since the volume V is arbitrary it follows that

$$/1-\Phi/\varrho_S \frac{d\vec{v}_S}{dt_S} + \alpha_F\Phi\varrho_F\frac{d\vec{v}_F}{dt_F} + \alpha_G\Phi\varrho_G\frac{d\vec{v}_G}{dt_G} =$$

$$= /1-\Phi/\varrho_S\vec{g}_S + \alpha_F\Phi\varrho_F\vec{g}_F + \alpha_G\Phi\varrho_G\vec{g}_G +$$

$$+ \mathrm{Div}\left[/1-\Phi/\,\underline{\underline{F}}_S + \alpha_F\Phi\,\underline{\underline{F}}_F + \alpha_G\Phi\,\underline{\underline{F}}_G\right].$$

This equation of motion is valid for any complex continuum, regardless of the form of the stress tensor and other particular properties of the material may take. This equation can't be reduced to three independent vector equation referring to the solid, liquid and the gas component. This is impossible in consequence of the interphase momentum transport. If we separate the terms referring to the rock, water and gas component, all others can be regarded as the interacting forces between the different phases. Thus we get

$$/1-\Phi/\varrho_S\frac{d\vec{v}_S}{dt_S} = /1-\Phi/\varrho_S\vec{g}_S + \mathrm{Div}\left[/1-\Phi/\,\underline{\underline{F}}_S\right] + \vec{k}_S$$

This is the equation of motion for the solid component of the complex continuum, where $\vec{k}_S$ is the interacting force between the solid and the other two components. Similarly we can obtain that

$$\alpha_F\Phi\varrho_F\frac{d\vec{v}_F}{dt_F} = \alpha_F\varrho_F\Phi\vec{g}_F + \mathrm{Div}\,/\alpha_F\Phi\underline{\underline{F}}_F/ + \vec{k}_F$$

and

$$\alpha_G\Phi\varrho_G\frac{d\vec{v}_G}{dt_G} = \alpha_G\Phi\varrho_G\vec{g}_G + \mathrm{Div}\,/\alpha_G\Phi\underline{\underline{F}}_G/ + \vec{k}_G\,.$$

Now we have nine further scalar equations, but there are three unknown extraneous forces in them. To obtain any solution we must postulate any simplified relationship for the unknown force $\vec{k}$. Furthermore it is approximately true that certain terms of the equations are negligible depending on the accompanying circumstances. For example the density of the gas phase is negligible compared with the fluid or the rock. In other hand the effect of tangential stresses is small in many practical cases for fluids, expecially for gases, however and therefore it is not unreasonable to consider the idealized situation in which the tangential stresses are neglected. Thus we get

$$\underline{\underline{F}}_F = - p_F \, \underline{\underline{I}}$$

and

$$\underline{\underline{F}}_G = - p_G \, \underline{\underline{I}}$$

where $\underline{\underline{I}}$ is the unit tensor, p is the pressure.

It is frequently convenient to neglect the rate of momentum of the solid phase. These simplifying assumptions allow to obtain approximate solutions in certain cases.

The principle of conservation of angular momentum is guaranteed, if the stress tensor is symmetric, i.e.

$$F^{ij} = F^{ji}.$$

If the stress tensor is symmetric, the following equation is not independent of the equation of motion:

$$\frac{d}{dt_S}\int_V \vec{r} \times /1 - \Phi/\varrho_S \vec{v}_S \, dV + \frac{d}{dt_F}\int_V \vec{r} \times \alpha_F \Phi \varrho_F \vec{v}_F \, dV + \frac{d}{dt_G}\int_V \vec{r} \times \alpha_G \Phi \varrho_G \vec{v}_G \, dV = \int_V \vec{r} \times \left[/1 - \Phi/\varrho_S \vec{g}_S + \alpha_F \Phi \varrho_F \vec{g}_F + \alpha_G \Phi \varrho_G \vec{g}_G \right] dV + \int_{/A/} \vec{r} \times \left[/1 - \Phi/ \underline{\underline{F}}_S + \alpha_F \Phi \underline{\underline{F}}_F + \alpha_G \Phi \underline{\underline{F}}_G \right] d\vec{A}$$

Let the kinetic energy of an arbitrary volume V of the complex continuum

$$\int_V \left[/1 - \Phi/\varrho_S \frac{v_S^2}{2} + \alpha_F \Phi \varrho_F \frac{v_F^2}{2} + \alpha_G \Phi \varrho_G \frac{v_G^2}{2} \right] dV,$$

and let the deformation tensor

$$\underline{\underline{S}} = \frac{1}{2} /\nabla \bullet \vec{v} + \vec{v} \bullet \nabla/ .$$

Then the rate of change of kinetic energy is equal to the rate at which work is being done on the volume by external forces, diminished by a dissipation term involving the interaction of stress and deformation which is transformed into heat:

$$\frac{d}{dt_S}\int_V /1 - \Phi/\varrho_S \frac{v_S^2}{2}\,dV + \frac{d}{dt_F}\int_V \alpha_F\Phi\varrho_F \frac{v_F^2}{2}\,dV +$$

$$\frac{d}{dt_G}\int_V \alpha_G\Phi\varrho_G \frac{v_G^2}{2}\,dV + \int_V \left[/1 - \Phi/\ \underline{\underline{S}}_S : \underline{\underline{F}}_S + \alpha_F\Phi\ \underline{\underline{S}}_F : \underline{\underline{F}}_F + \right.$$

$$\left. + \alpha_G\Phi\ \underline{\underline{S}}_G : \underline{\underline{F}}_G\right] dV = \int_V \left[/1 - \Phi/\varrho_S\, \vec{g}_S\, \vec{v}_S + \alpha_F\Phi\varrho_F\, \vec{g}_F\, \vec{v}_F + \right.$$

$$\left. + \alpha_G\Phi\varrho_G\, \vec{g}_G\, \vec{v}_G\right] dV + \int_{/A/} \left[/1 - \Phi/\ \vec{v}_S\, \underline{\underline{F}}_S + \alpha_F\Phi\ \vec{v}_F\, \underline{\underline{F}}_F + \right.$$

$$\left. + \alpha_G\Phi\ \vec{v}_G\, \underline{\underline{F}}_G\right] d\vec{A}$$

This scalar equation can also be written in differential form:

$$/1 - \Phi/\varrho_S \frac{d}{dt_S}\,/\frac{v_S^2}{2}/ + \alpha_F\Phi\varrho_F \frac{d}{dt_F}\,/\frac{v_F^2}{2}/ + \alpha_G\Phi\varrho_G \frac{d}{dt_G}\,/\frac{v_G^2}{2}/ +$$

$$+ /1 - \Phi/\ \underline{\underline{S}}_S : \underline{\underline{F}}_S + \alpha_F\Phi\ \underline{\underline{S}}_F : \underline{\underline{F}}_F + \alpha_G\Phi\ \underline{\underline{S}}_G : \underline{\underline{F}}_G =$$

$$= /1 - \Phi/\varrho_S\, \vec{g}_S\, \vec{v}_S + \alpha_F\Phi\varrho_F\, \vec{g}_F\, \vec{v}_F + \alpha_G\Phi\varrho_G\, \vec{g}_G\, \vec{v}_G +$$

$$+ \mathrm{Div}\left[/1 - \Phi/\ \underline{\underline{F}}_S\, \vec{v}_S + \alpha_F\Phi\ \underline{\underline{F}}_F\, \vec{v}_F + \alpha_G\Phi\ \underline{\underline{F}}_G\, \vec{v}_G\right.$$

Finally the principle of conservation of total energy of a volume V of the complex continuum can be written. We define the total energy of a volume V as the sum of its kinetic energy and its internal energy E. The thermodinamical interpretation of the internal energy is different for compressible and incompressible media. For a compressible material for example gases we assume that internal energy is a thermodynamic state variable, which satisfies the following equation

$$TdS = dE + p\, d \,/\frac{1}{\varrho}/$$

where T is the absolute temperature, S is the entropy per unit volume, p is the pressure. For incompressible materials the pressure is not a thermodynamical variable, because the density is constant we obtain a simpler equation:

$$TdS = dE$$

The principle of conservation of the total energy states that the rate of increase of the total energy of a material volume V is equal the rate at which work is being done on the volume plus the rate at which heat is conducted into the volume.

$$\frac{d}{dt_S} \int_V /1-\Phi/\varrho_S \,/\frac{v_S^2}{2} + E_S/\, dV + \frac{d}{dt_F} \int_V \alpha_F \varrho_F \Phi \,/\frac{v_F^2}{2} + E_F/\, dV +$$

$$+ \frac{d}{dt_G} \int_V \alpha_G \Phi \varrho_G \,/\frac{v_G^2}{2} + E_G/\, dV = \int_V \Big[/1-\Phi/\varrho_S \vec{g}_S \vec{v}_S -$$

$$+ \alpha_F \Phi \varrho_F \vec{g}_F \vec{v}_F + \alpha_G \Phi \varrho_G \vec{g}_G \vec{v}_G\Big] dV + \int_{/A/} \Big[/1-\Phi/\vec{v}_S \underline{\underline{F}}_S +$$

$$+ \alpha_G \Phi \vec{v}_G \underline{\underline{F}}_G \Big] d\vec{A} + \int_{/A/} \Big[\lambda_S \, \varepsilon_S \, \mathrm{grad} \, T_S + \lambda_F \, \varepsilon_F \, \mathrm{grad} \, T_F +$$

$$+ \lambda_G \, \varepsilon_G \, \mathrm{grad} \, T_G \Big] d\vec{A}$$

Applying the divergence theorem to the surface integrals we can get the differential form of the total energy equation, referring to the unit volume of the complex continuum:

$$/1-\Phi/ \varrho_S \frac{d}{dt_S} /\frac{v_S^2}{2} + E_S/ + \alpha_F \Phi \varrho_F \frac{d}{dt_F} /\frac{v_F^2}{2} + E_F/ +$$

$$+ \alpha_G \Phi \varrho_G \frac{d}{dt_G} /\frac{v_G^2}{2} + E_G/ = /1-\Phi/ \varrho_S \vec{g}_S \vec{v}_S + \alpha_F \Phi \varrho_F \vec{g}_F \vec{v}_F +$$

$$+ \alpha_G \Phi \varrho_G \vec{g}_G \vec{v}_G + \mathrm{div} \Big[/1-\Phi/ \underline{\underline{F}}_S \vec{v}_S + \alpha_F \Phi \underline{\underline{F}}_F \vec{v}_F +$$

$$+ \alpha_G \Phi \underline{\underline{F}}_G \vec{v}_G + \mathrm{div} \Big[/1-\Phi/ \lambda_S \, \mathrm{grad} \, T_S + \alpha_F \Phi \lambda_F \, \mathrm{grad} \, T_F +$$

$$+ \alpha_G \Phi \lambda_G \, \mathrm{grad} \, T_G \Big]$$

The basic equations of the dynamics of complex continua are discussed.

The number of unknown variables is more in these equations than the number of independent scalar equations. Thus we need further equations to solve any problem. In general one may adjoin to these equations a thermodinamical relation, the equation of state:

$$p = f\,/\varrho,\ T/\ .$$

This equation may be different for gases, liquids or solids. The pressure and temperature dependence of density especially complex for multicomponent materials. With few exceptions, data are insufficient to make generalizations other than that methods suitable for pure materials may be applied with approximate results.

The other necessary equation is the so-called constitutive equation. It means the relation between the stress tensor and other kinematic variables. The stress tensor governs the dynamic response of the medium, by relating it to other kinematic and thermodynamic variables we can define the type of medium with which we work.

The most primitive form of the constitutive equation refers to inviscid liquids or gases

$$\underline{\underline{F}} = -p\,\underline{\underline{I}}\ .$$

Stokes derived a constitutive equation applicable to fluids which exert appreciable tangential stresses:

$$\underline{\underline{F}} = \left[-p + \varkappa \operatorname{div}\vec{v} + \varkappa^{*}/\operatorname{div}\vec{v}/^{2}\right]\underline{\underline{I}} + /2\mu + 2\mu^{*}\operatorname{div}\vec{v}/\,\underline{\underline{S}} + {}$$
$$+\,4\nu\,\underline{\underline{S}}^{2}$$

which is simpler for incompressible fluids:

$$\underline{\underline{F}} = -p\,\underline{\underline{I}} + 2\mu\,\underline{\underline{S}} + 4\nu\,\underline{\underline{S}}^{2}\ .$$

Here μ is the viscosity, $\varkappa$, $\varkappa^{*}$, μ^{*}, ν are further constants.

The most general constitutive equation is presented by Asszonyi. He derived the following equation based on Onsager's law:

$$\underline{\underline{F}} = \underline{\underline{\Lambda}} \cdot /\nabla \bullet \vec{v}/$$

where $\underline{\underline{\Lambda}}$ is the conduction tensor of momentum. Its matrix is fourth-order, it has $3^4 = 81$ scalar elements. Regarding the symmetric properties of the tensor $\underline{\underline{\Lambda}}$, it has 21 independent elements in it. This is valid for perfectly anisotropic media.

If the conductive terms are dominated in the transport of momentum, the constitutive equation is the following

$$\underline{\underline{F}} = \underline{\underline{\Lambda}} \cdot /\nabla \bullet \vec{u}/$$

where $\vec{u}$ is the displacement vector.

The constitutive equation for the complex continuum can be the linear combination the above two equations:

$$\underline{\underline{F}} = /1-\Phi/\ \underline{\underline{F}}_S + \alpha_F \Phi\ \underline{\underline{F}}_F + \alpha_G \Phi\ \underline{\underline{F}}_G =$$

$$= /1-\Phi/ \underline{\underline{\Lambda}}_S \cdot /\nabla \bullet \vec{u}/ + \alpha_F \Phi \underline{\underline{\Lambda}}_F \cdot /\nabla \circ \vec{v}_F/ +$$

$$+ \alpha_G \Phi \underline{\underline{\Lambda}}_G /\nabla \bullet \vec{v}_G/$$

Thus the system of basic equations is complete. It includes many special cases.

If $\dfrac{\partial \Phi}{\partial t} = 0$, $\alpha_F = 1$, $\alpha_G = 0$, we get the equations for seeping flow through a rigid, porous media. The $\alpha_F = 0$,

$\alpha_e = 1$ case refers to the seepage of the gas. Allowing that $\frac{\partial \Phi}{\partial t} \neq 0$, the equations of consolidation can be derived.

The basic equation of hydraulic and pneumatic transport also can be deduced from this basic equation system.

We have the intention to introduce a general fenomenological treatment for the multicomponent systems based on continuum mechanics. We hope a better understanding of this very important topics for the practice of mining.

REFERENCES

[1] ASSZONYI, CS. and RICHTER, R.: The Rheological Theory of Rock Mechanics /in Hungarian/.
NIMTK, Budapest /1975/.

[2] ERINGEN, C.A.: Nonlinear theory of Continuous Media.
Mc.Graw Hill, New York /1962/.

[3] JAEGER, J.C. and COOK, N.G.W.: Fundamentals of Rock Mechanics.
Chapman and Hall, London /1976/.

[4] MUSKHELISVILI, N.I.: Some Basic Problems of the Mathematical Theory of Elasticily.
Noordhoff, Groningen /1933/.

[5] ONSAGER, L.: Phys. Rev. 37, 405 /1931/.